# Automorphisms of the Algebra of Fuzzy Truth Values II

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**Abstract**

One special subalgebra of the algebra of fuzzy truth values is the set of those elements that take only 0 and 1 as values. This subalgebra is in one-to-one correspondence with the subsets of the unit interval, but the operations do not correspond to ordinary union and intersection. The automorphisms of this subalgebra are studied here, resulting in showing that the subobjects corresponding to finite subsets is a characteristic subalgebra of the algebra of fuzzy truth values, and that the subalgebra corresponding to closed intervals is also a characteristic subalgebra. A number of facts about the automorphisms of the subalgebra in question are proved, but we have not yet succeeded in showing that it is a characteristic subalgebra of the algebra of fuzzy truth values.

**Keywords**: Type-2 fuzzy sets, automorphisms

## 1. The Algebra of Sets in $\mathbb{M}$

We refer the reader to the previous paper *Automorphisms of the Algebra of Fuzzy Truth Values I*, [2], for notation and terminology. It is easy to verify that

$$\mathbb{S} = (\{0,1\}^{[0,1]}, \sqcup, \sqcap, 0, 1)$$

is a subalgebra of $\mathbb{M} = ([0,1]^{[0,1]}, \sqcup, \sqcap, \bar{0}, \bar{1})$. This subalgebra we call the **algebra of sets in** $\mathbb{M}$. Its elements are in one-to-one correspondence with the subsets of $[0,1]$, but the operations do not correspond to ordinary union and intersection of sets. A basic question is whether automorphisms of $M$ induce automorphisms of $\mathbb{S}$, that is, whether or not $\mathbb{S}$ is a characteristic subalgebra of $\mathbb{M}$. Note that $\mathbb{S}$ contains the subalgebra $\mathbb{I}$ of singletons, and the subalgebra $\mathbb{I}^{[2]}$ of closed intervals. Also the set $\mathbb{F}$ of finite sets in $\mathbb{M}$, that is the elements of $\mathbb{S}$ that have value 1 at only finite number of elements of $[0,1]$, is a subalgebra of $\mathbb{S}$. Thus we have the picture

$$\begin{tikzcd} & \mathbb{M} & \\ & \mathbb{S} \arrow[u] & \\ \mathbb{F} \arrow[ur] & & \mathbb{I}^{[2]} \arrow[ul] \\ & \mathbb{I} \arrow[ul] \arrow[ur] & \\ & \{\bar{0},\bar{1}\} \arrow[u] & \end{tikzcd}$$

of subalgebras of $\mathbb{M}$, and we know that $\mathbb{I}$, the subalgebra of singletons, is a characteristic subalgebra of $\mathbb{M}$. Our goal is to show that $\mathbb{S}$ is characteristic. We have not been able to do that, but in endeavoring to do so, will get that $\mathbb{F}$ and $\mathbb{I}^{[2]}$ are characteristic and some information about the action on $\mathbb{S}$ of automorphisms of $\mathbb{M}$

From [1], we have the following for any automorphism $\varphi$ of $\mathbb{M}$.

- $\varphi(\bar{0}) = \bar{0}$ and $\varphi(\bar{1}) = \bar{1}$
- $\varphi(\underline{1}) = \underline{1}$ and $\varphi(\underline{0}) = \underline{0}$
- $\varphi(\overline{a})$ is a singleton for any singleton $\overline{a}$
- $\varphi(\{0,1\}) = \{0,1\}$
- $\varphi(f^L) = \varphi(f)^L$; $\varphi(f^R) = \varphi(f)^R$ for any $f \in \mathbb{M}$.

We will, as in [2], omit proofs which will appear later in an expanded paper. The principal fact we develop is Theorem 6 below.

*Lemma 1:* For $a \in [0,1]$ and $\varphi \in \mathit{Aut}(\mathbb{M})$, $\varphi(\overline{0} \vee \overline{a}) = \overline{0} \vee \varphi(\overline{a})$ and $\varphi\left(\overline{a} \vee \overline{1}\right) = \varphi\left(\overline{a}\right) \vee \overline{1}$.

If $f$ and $g$ are sets, we say $f$ **is to the left of** $g$ and write $f \ll g$ if $f(x) > 0$ implies $g(y) = 0$ for all $y < x$. Similarly, we say $g$ **is to the right of** $f$ and write $g \gg f$ if $g(x) > 0$ implies $f(y) = 0$ for all $y > x$. This definition leaves open the possibility that $f$ and $g$ are both nonzero at at most one point. Note that $f \ll g$ if and only if $g \gg f$. For singletons $\overline{a}$, it is clear that $\overline{a} \ll g$ if and only if $\overline{a}^L \wedge g = g$. If $g$ is a set, this is equivalent to the condition that $\overline{a} \wedge g^R = \overline{a}$.

*Proposition 2:* Let $a \in [0,1]$ and let $g \in \mathbb{M}$. Then $\overline{a} \ll g$ if and only if $\overline{a} \sqcup g = g$, and $\overline{a} \gg g$ if and only if $\overline{a} \sqcap g = g$.

We need to know when automorphisms preserve the property of being to the left or right. In the following corollary, we use the fact that $\varphi(\overline{a})$ is a singleton.

*Corollary 3:* $\overline{a} \ll g$ if and only if $\varphi(\overline{a}) \ll \varphi(g)$, and $\overline{a} \gg g$ if and only if $\varphi(\overline{a}) \gg \varphi(g)$.

It follows immediately from this corollary that if $\overline{a}$ is not to the right of $g$, then for any automorphism $\varphi$ of $\mathbb{M}$, $\varphi(\overline{a})$ is not to the right of $\varphi(g)$. And if $\overline{a}$ is not to the left of $g$, then for any automorphism $\varphi$ of $\mathbb{M}$, $\varphi(\overline{a})$ is not to the left of $\varphi(g)$. The properties of being not to the right and not to the left can be characterized as follows.

*Lemma 4:* Let $a \in [0,1]$ and $g \in \mathbb{S}$. If $\overline{a}$ is not to the right of $g$, then $(\overline{a} \vee \overline{1}) \sqcap g = \overline{a} \vee g$. If $\overline{a}$ is not to the left of $g$, then $(\overline{0} \vee \overline{a}) \sqcup g = \overline{a} \vee g$.

*Lemma 5:* If $f \in \mathbb{M}$ is normal, then $f^L \vee f^R = \underline{1}$.

*Theorem 6:* Let $a \in [0,1]$, $g \in \mathbb{S}$, and $\varphi \in Aut(\mathbb{M})$. Then $\varphi(\overline{a} \vee g) = \varphi(\overline{a}) \vee \varphi(g)$.

By induction, we have the following.

*Corollary 7:* If $g$ is a finite set in $\mathbb{M}$, then $\varphi(g)$ is a finite set with the same number of elements. In fact, if $g = \overline{a_1} \vee \overline{a_2} \vee \cdots \vee \overline{a_n}$ then $\varphi(g) = \varphi(\overline{a_1}) \vee \varphi(\overline{a_2}) \vee \cdots \vee \varphi(\overline{a_n})$.

*Corollary 8:* The algebra $\mathbb{F}$ of finite subsets of $\mathbb{M}$ is a characteristic subalgebra of $\mathbb{M}$.

We now look at the algebra $\mathbb{I}^{[2]}$ of intervals in $\mathbb{M}$. Suppose $g = [a,b]$. Then $\overline{a} \ll g$ and $\overline{b} \gg g$. From above, we see that $\varphi(\overline{a}) \ll \varphi(g) \ll \varphi(\overline{b})$. Moreover, $g = \overline{a} \vee g = \overline{b} \vee g$ so we know that $\varphi(g) = \varphi(\overline{a}) \vee \varphi(g) = \varphi(\overline{b}) \vee \varphi(g)$. Let $\overline{c} = \varphi(\overline{a})$ and $\overline{d} = \varphi(\overline{b})$. Then $\varphi(g)(c) = \varphi(g)(d) = 1$. But $\varphi(g)$ is a convex function. All this together yields the following.

*Theorem 9:* Let $\varphi \in Aut(M)$ and $a \leq b$ in $[0,1]$. If $\varphi(\overline{a}) = \overline{c}$ and $\varphi(\overline{b}) = \overline{d}$, then $\varphi([a,b]) = [c,d]$.

*Corollary 10:* The algebra $\mathbb{I}^{[2]}$of intervals in $\mathbb{M}$ is a characteristic subalgebra of $\mathbb{M}$.

# 2. Comments

We have not been able yet to prove that $\mathbb{S}$ is a characteristic subalgebra of $\mathbb{M}$. If $g \in \mathbb{S}$, then $g = \vee_{a \in S}\bar{a}$ for some subset $S$ of $[0,1]$. For any automorphism $\varphi$ of $\mathbb{M}$, $\varphi(g) = \varphi(\vee_{a \in S}\bar{a}) \leq (\vee_{a \in S}\varphi(\bar{a}))$. We need this inequality to be an equality, but have not been able to prove that yet, although we conjecture that equality holds.

The map $\mathbb{M} \to \mathbb{S}$ that takes an element $g$ to the element that is $1$ when $g > 0$ and $0$ otherwise is a projection. The proof is straightforward. The equivalence relation is to identify two elements if they are $> 0$ at the same places. This is a congruence. This fact should be useful in studying $\mathbb{S}$.

The set of closed intervals forms a characteristic subalgebra. We have shown that $\mathbb{I}$ and $\mathbb{L}$ are characteristic subalgebras of $\mathbb{M}$. Automorphisms of $\mathbb{M}$ carry monotonic functions to monotonic functions of the same type.

## References


[1] Walker, C., and Walker, E., The Algebra of Fuzzy Truth Values, *Fuzzy Sets and Systems*, 149(2) (2005), 309-347.

[2] Walker, C., and Walker, E., Automorphisms of the Algebra of Fuzzy Truth Values I, *Proceedings of JCIS 2005, Salt Lake City, Utah, July 21-26, 2005*.