Sufficient Conditions for Linear Dependency and Independency of Interval Vectors

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***Abstract*— To the best knowledge of authors, none of existing literatures is available for the linear dependency and independency test of interval vectors. In this paper, the definitions about the linear dependency and independency of interval vectors are given, and an effective way for checking the linear dependency and independency of interval vectors is suggested.**

***Index Terms*— Linear dependency, Linear independency, Interval vectors**

I. Introduction

In information science, reliable computation and/or robust control area, the model uncertainty problem has been effectively and popularly handled by "interval" concept. Great amount of literatures are available under the name of "interval" for example, interval algebra [1], [2], interval polynomial [3], [4], Schur stability of interval matrices [5], [6], Hurwitz stability of interval matrices [7], [8], interval polynomial matrices [9], eigenvalues of interval matrices [10], [11], and robust control with parameter uncertainty [12], [13]. However, the property of interval vectors has not been fully studied. In fact, the basic concepts of interval vectors were defined in [1], [2], and as a specified example of the quasivector spaces, the interval vectors have been defined in [14]. Particularly, the algebraic property of the interval vectors was studied in [15], but in these works, even though the linear combination of interval vectors has been studied, the linear dependency and independency conditions were not investigated.

In this paper, we define the linear (in)dependency of interval vectors. However overall mathematical property of our definition is same to [15], only our definition is more or less based on linear algebra, while definitions given in [15] are based on the theory of the quasilinear spaces with group structure.

This paper consists of as follows: In Section II, we provide sufficient linear dependency and independency conditions of the interval vectors. Conclusions and possible applications in information sciences are discussed in Section III.

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II. Linear Dependency and Independency of Interval Vectors

Throughout this paper, we use the following basic definitions. Our discussions are limited to the real space.

Definition 2.1: A real interval scalar x^I is defined as: $x^I := [\underline{x}, \overline{x}]$, where $\underline{x}, \overline{x} \in \mathcal{R}$ and for all $x \in x^I$, there exists a corresponding λ such that $x = \lambda\overline{x} + (1-\lambda)\underline{x}$ with $0 \le \lambda \le 1$, $\lambda \in \mathcal{R}$. The n-dimensional real column interval vector $\mathbf{x}^I$ is defined as: $\mathbf{x}^I := (x_1^I, \ldots, x_n^I)^T$ and the $n \times m$ dimensional real interval matrix is defined from the interval vectors as:

$$X^I := \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \ldots, \mathbf{x}_m^I\right)$$

The interval vector and interval matrix can be written as: $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and $X^I = \left[\underline{X}, \overline{X}\right]$. Or, they can be written as: $\mathbf{x}^I = [\mathbf{x}_0 - \Delta\mathbf{x}, \mathbf{x}_0 + \Delta\mathbf{x}]$ and $X^I = [X_0 - \Delta X, X_0 + \Delta X]$, where $\mathbf{x}_0 = \frac{\overline{\mathbf{x}}+\underline{\mathbf{x}}}{2}$, $X_0 = \frac{\overline{X}+\underline{X}}{2}$, $\Delta\mathbf{x} = \frac{\overline{\mathbf{x}}-\underline{\mathbf{x}}}{2}$, and $\Delta X = \frac{\overline{X}-\underline{X}}{2}$.

Based on [1], [2], the following interval arithmetics are used in this paper.

Definition 2.2: The intersection of two real interval scalars x^I and y^I is defined as: $x^I \cap y^I := \left\{z \mid z \in x^I \text{ and } z \in y^I\right\}$. The union of two real interval scalars x^I and y^I is defined as: $x^I \cup y^I := \left\{z \mid z \in x^I \text{ or } z \in y^I\right\}$.

Definition 2.3: The addition of two real interval scalars x^I and y^I is defined and calculated as: $x^I \oplus y^I = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right]$, the substraction is $x^I \ominus y^I = \left[\underline{x} - \overline{y}, \overline{x} - \underline{y}\right]$, and the multiplication is

$$x^I \otimes y^I = \left[\min\left\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right\}, \max\left\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right\}\right]$$

The division should be carefully defined as [2]:

$$\begin{aligned} \frac{1}{x^I} &= \emptyset \text{ if } x^I = [0,0] \\ &= \infty \text{ if } x^I = (0, 0^+] \\ &= -\infty \text{ if } x^I = [0^-, 0) \\ &= \left[\frac{1}{\overline{x}}, \frac{1}{\underline{x}}\right] \text{ if } x^I > 0 \\ &= \left[\frac{1}{\underline{x}}, \frac{1}{\overline{x}}\right] \text{ if } x^I < 0 \\ &= [-\infty, \infty] \text{ if } \underline{x} < 0 \text{ and } \overline{x} > 0 \end{aligned} \qquad (1)$$

Then, the division of two interval scalars is simply defined and calculated as: $x^I \oslash y^I = x^I \otimes \frac{1}{y^I}$.

Definition 2.4: The *ratio* $\mathbf{r}_{xy}$ between two interval vectors is defined and calculated as:

$$\mathbf{r}_{xy} := \mathbf{x}^I \setminus \mathbf{y}^I = \left(x_1^I \oslash y_1^I, \ldots, x_n^I \oslash y_n^I\right)$$

The addition, substraction, dot-product, and cross-product of two interval vectors and interval matrices can be defined based on above scalar arithmetics.

The interval arithmetics of a real interval scalar by itself should be distinguished from the arithmetics of two different scalar intervals. For the linear time invariant (LTI) system [1], we use the following definitions:

Definition 2.5: If x^I is not time dependent (i.e., time invariant), the addition of a real interval scalar x^I by itself is defined and calculated as: $x^I \oplus x^I = [\underline{x}+\underline{x}, \overline{x}+\overline{x}]$, the substraction is $x^I \ominus x^I = 0$, and the multiplication is $x^I \otimes x^I = \left[\alpha^2, \beta^2\right]$, where $\alpha = \min\{|\underline{x}|, |\overline{x}|\}$; $\beta = \max\{|\underline{x}|, |\overline{x}|\}$. The division is defined as: $x^I \oslash x^I = 1$ if $x^I \neq [0,0]$.

In linear algebra, the following linear (in)dependency condition of the linear vectors is popularly used.

Definition 2.6: Without interval, when n different vectors are given as: $\mathbf{x}_1, \ldots, \mathbf{x}_n$, they are called *linearly independent* iff there exist only trivial solutions ($a_1 = a_2 = \cdots = a_n = 0$) such that $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = 0$. Otherwise, they are *linearly dependent*. If they are linearly independent, any $\mathbf{x}_i$ cannot be produced by any combinations of other vectors.

Now, with the basic definitions given above, we define the linear (in)dependency of interval vectors based on Definition 2.6.

Definition 2.7: With interval, let us suppose we have n different interval column vectors given as: $\mathbf{x}_1^I, \ldots, \mathbf{x}_n^I$. They are called *linearly independent* iff there exist only trivial solutions ($a_1 = a_2 = \cdots = a_n = 0$) such that $a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I \oplus \cdots \oplus a_n\mathbf{x}_n^I = 0$. Otherwise, we say that the interval vectors are *linearly dependent*.

Before considering the general case, let us first consider the linear (in)dependency of two interval vectors. Supposing that two interval vectors are given as: $\mathbf{x}_1^I$ and $\mathbf{x}_2^I$, and based on Definition 2.7, two interval vectors are linearly independent iff there exist only trivial solutions $a_1 = a_2 = 0$ such that

$$a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I = 0. \tag{2}$$

Here, notice that it is not easy to get solutions for (2) directly. However, if we use "ratio" concept, we can check the linear (in)dependency property easily, which is expressed in the following theorem:

Theorem 2.1: Two n dimensional LTI interval vectors $\mathbf{x}^I, \mathbf{y}^I$ with $0 \notin x_1^I \cap x_2^I \cap \ldots \cap x_n^I$, $0 \notin y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, are linearly independent iff, from the ratio $\mathbf{r}_{xy}$ of $\mathbf{x}^I, \mathbf{y}^I$, the following equality holds:

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n = \emptyset, \tag{3}$$

where $(\mathbf{r}_{xy})_i$ can be defined as $x_i^I \oslash y_i^I$.

Proof: Sufficiency: From $a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I = 0$, we have

$$a_1\left[x_1^I, x_2^I, \ldots, x_n^I\right]^T = \ominus a_2\left[y_1^I, y_2^I, \ldots, y_n^I\right]^T \tag{4}$$

¿From Definition 2.4 and Definition 2.5, and using the commutative and associative property of interval scalars, the ratio of each elements are

$$\begin{aligned} x_i^I \oslash y_i^I &= (\mathbf{r}_{xy})_i \\ \Leftrightarrow x_i^I \otimes \frac{1}{y_i^I} &= (\mathbf{r}_{xy})_i \\ \Leftrightarrow x_i^I \otimes \frac{1}{y_i^I} \otimes y_i^I &= (\mathbf{r}_{xy})_i \otimes y_i^I \\ \Leftrightarrow x_i^I &= (\mathbf{r}_{xy})_i \otimes y_i^I \end{aligned} \tag{5}$$

By inserting (5) to the left-hand side of (4), the followings are true:

$$\begin{aligned} & a_1\left[(\mathbf{r}_{xy})_1 \otimes y_1^I, (\mathbf{r}_{xy})_2 \otimes y_2^I, \ldots, (\mathbf{r}_{xy})_n \otimes y_n^I\right]^T = \\ & \qquad \ominus a_2\left[y_1^I, y_2^I, \ldots, y_n^I\right]^T \\ \Leftrightarrow\ & a_1\left[(\mathbf{r}_{xy})_1, (\mathbf{r}_{xy})_2, \ldots, (\mathbf{r}_{xy})_n\right]^T = -a_2\left[1, 1, \ldots, 1\right]^T \\ \Leftrightarrow\ & \left[(\mathbf{r}_{xy})_1, (\mathbf{r}_{xy})_2, \ldots, (\mathbf{r}_{xy})_n\right]^T = -\frac{a_2}{a_1}\left[1, 1, \ldots, 1\right]^T \end{aligned} \tag{6}$$

Here, from (6), we have

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n = -\frac{a_2}{a_1}, \tag{7}$$

so, $(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n = \emptyset$, we have $\frac{a_2}{a_1} = \emptyset$. Thus, since only $a_1 = 0$ is the solution for $\frac{a_2}{a_1} = \emptyset$, and $0 \notin x_1^I \cap x_2^I \cap \ldots \cap x_n^I$ and $0 \notin y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, we have $a_2 = 0$.

Necessity: Let us suppose that

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n \neq \emptyset,$$

then we can have $a_2 = 0$ and $a_1 \neq 0$, or $a_2 \neq 0$ and $a_1 \neq 0$. Thus, by definition, this is not linearly independent. ■

Let us further think the case $0 \in x_1^I \cap x_2^I \cap \ldots \cap x_n^I$ or $0 \in y_1^I \cap y_2^I \cap \ldots \cap y_n^I$.

Theorem 2.2: If $0 \in x_1^I \cap x_2^I \cap \ldots \cap x_n^I$ or $0 \in y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, two interval vectors are then linearly dependent.

Proof: With any a_1 and $a_2 = 0$, or with $a_1 = 0$ and any a_2, the following equality can be true:

$$a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I = 0.$$

So, By Definition 2.7, the proof is completed. ■

Although above theorems are effective for checking the linear (in)dependency of two interval vectors, it is difficult to extend above theorems to more than three interval vectors. Let us suppose that we have three different interval vectors,

[1]For linear time varying case, we have to use Definition 2.3.

which are given as: $\mathbf{x}^I, \mathbf{y}^I, \mathbf{z}^I$ and we want to check the linear (in)dependency of them. The first task is to check the linear dependency between two interval vectors. This task can be performed from preceding results, but we also have to check the linear combination case. That is, we have to check if there exist trivial solutions $a_1 = a_2 = a_3 = 0$ such that

$$a_1\mathbf{x}^I \oplus a_2\mathbf{y}^I \oplus a_3\mathbf{z}^I = 0.$$

It looks quite tough to solve this simple equation, furthermore our ultimate goal is to find the general case such as:

$$a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I \oplus \cdots \oplus a_3\mathbf{x}_n^I = 0.$$

So, apparently, it is almost impossible to check the linear (in)dependency of the interval vectors [2]. In the sequel, we suggest one simple but very effective sufficient condition for checking the linear (in)dependency of interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ where an $\mathbf{x}_i^I$ is an interval vector in $\mathcal{R}^m$. For the accurate description of our idea, we separately consider three different cases.

$$\text{Case}-1: m > n. \ \ \text{Case}-2: m = n. \ \ \text{Case}-3: m < n$$

We only investigate Case-1. In fact, Case-2 and Case-3 can be investigated using the analysis of Case-1. For convenience, the following concepts are necessary. In the following $m \times n,\ m > n$, matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ m_{31} & m_{32} & \dots & m_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m1} & m_{m2} & \dots & m_{mn} \end{bmatrix}$$

let us select whole possible $n \times n$ sub-matrices, which are expressed as:

$$S^i = \begin{bmatrix} s_{11}^i & s_{12}^i & \dots & s_{1n}^i \\ s_{21}^i & s_{22}^i & \dots & s_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1}^i & s_{n2}^i & \dots & s_{nn}^i \end{bmatrix},$$

where $i = 1, \dots, k$. Then, it is easy to notice that the total number of possible sub-matrices S^i is calculated by:

$$k = \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$$

Sub-matrices S^i are composed of n different row vectors of M. The index of n different row vectors of S^i is represented by a set such as: $s^i = \{\text{index of row vectors of } M\},\ i = 1, \dots, k$. For the accurate translation of our idea, we make a definition as follows:

Definition 2.8: In this paper, we call sub-matrices $S_M = \{S^i,\ i = 1, \dots, k\}$ as *square set* and S^i as *sub-square matrices*, and $s_M = \{s^i,\ i = 1, \dots, k\}$ is called *index set* and s^i is called *index*.

Now, we consider the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$. Let us write these interval vectors in an interval matrix form such as:

$$X^I := \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I\right) \tag{8}$$

Then, X^I is an $m \times n$ interval matrix, so based on Definition 2.8, the corresponding square set of X^I can be found as: $S_X = \{S^i,\ i = 1, \dots, k\}$ where $k = \begin{pmatrix} m \\ n \end{pmatrix}$, and the corresponding index set of X^I can be found as $s_X = \{s^i,\ i = 1, \dots, k\}$. Here, we further define the center square matrices S_{X_c} and calculate them as:

$$S_{X_c} = \left\{ S_0^i = \frac{\underline{S^i} + \overline{S^i}}{2},\ i = 1, \dots, k \right\},$$

and define the radius square matrices ΔS_X and calculate them as:

$$\Delta S_X = \left\{ \Delta S^i = \frac{\overline{S^i} - \underline{S^i}}{2},\ i = 1, \dots, k \right\}$$

For our main result, notating the absolute value of a matrix A by $|A| = (|a_{ij}|)$, the following lemma can be adopted from [16].

Lemma 2.1: For interval matrix X^I, let its center matrix X_0 be nonsingular and the spectral radius $\rho\left(\left|(X_0)^{-1}\right| \Delta X\right) < 1$, then X^I is nonsingular.

Now, for the linear independency test of the interval vector set, we suggest the following theorem:

Theorem 2.3: For $S^I \in S_X$, if there exists at least one corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$ such that S_0 is nonsingular and $\rho\left(\left|(S_0)^{-1}\right| \Delta S\right) < 1$, then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent.

Proof: Let us consider $X^I = \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I\right)$, which is an $m \times n,\ m > n$, interval matrices composed of the interval vectors. It is a fact that the column vectors are linearly independent if (and only if in the point of "rank") the rank of X^I is n. Also from the fact that the row rank is equal to the column rank, so if S^I has rank n, then the column rank of X^I is also n. Therefore, if any one of $S^I \in S_X$ has row rank n, then X^I has n column rank. So, by Lemma 2.1, for S_0 and ΔS corresponding to S^I, if S_0 is nonsingular and $\rho\left(\left|(S_0)^{-1}\right| \Delta S\right) < 1$, then X^I has full column rank, because the nonsingular condition is equivalent to the full rank condition. Thus, since the full column rank indicates the linear independency, the proof is completed. ■

Remark 2.1: Theorem 2.3 checks the linear independency of the interval vector set using finite interval matrices set. The key idea of Theorem 2.3 is to investigate the linear independency of the interval vectors on the form of interval matrix. Using the fact that the row rank is equal to column rank and the full rank condition is equivalent to the linear

[2] As far as authors are concerned, nobody has suggested this kind of questions and there is no existing solution.

independency condition, Theorem 2.3 easily checks the linear independency of the interval vectors.

Using the proof of Theorem 2.3 and using the results of [16], we also can find the sufficient condition for linear dependency of the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$. Let us use the following lemma for this purpose.

Lemma 2.2: For interval matrix X^I, there exist a matrix R and a natural number p such that, element-wisely,

$$(I + |I - X_0 R|)_p \leq (\Delta X \, |R|)_p$$

where $p \in \{1, \ldots, n\}$ and $(\cdot)_p$ represents p^{th} column, then interval matrix X^I is singular.

Proof: See theorem 3.3 of [16]. ■

Theorem 2.4: For all $S^I \in S_X$ and for all its corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$, if there exist a matrix R and a natural number p such that, element-wisely,

$$(I + |I - S_0 R|)_p \leq (\Delta S \, |R|)_p \,,$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly dependent.

Proof: Theorem 2.3 shows that the interval vectors are linearly independent if there exists at least one S^I such that the conditions of Theorem 2.3 hold. So, to eliminate the case of Theorem 2.3, we have to check all $S^I \in S_X$ for the linearly dependent test. Hence, by checking all S^I and based on the proof of Theorem 2.3 and Lemma 2.2, the proof of Theorem 2.4 can be completed. ■

Next, let us consider Case-3 which is $m < n$ with interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ and $\mathbf{x}_i^I \in \mathcal{R}^m$. This problem is dual to Case-1, because, in this case, we can also define "square set" and "index set" as done in Case-1. Then using the same procedure as performed in Theorem 2.3 and in Theorem 2.4, the linear independency and dependency can be checked. However, in Case-3, it is recommended checking the linear dependency of two column vectors using Theorem 2.1 first. This approach will save the computational amount. In Case-2, since X^I is an interval matrix, without making square set and index set, Case-1 results can be directly utilized.

III. Conclusions

In this paper, we suggested the concept of "linear dependency" and "linear independency" of interval vectors. For the linear dependency and independency test, we suggested using the regularity property of the interval matrix. As potential applications of our result, the robust controllability, robust observability, stability analysis, and null-space of interval operators can be considered.

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IV. References

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