Dislocated Fuzzy Metric Spaces And Associated Fuzzy Topologies

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Abstract : The concept of dislocated fuzzy metric space is introduced and the associated fuzzy topologies are discussed. Generalized fuzzy versions of the Banach contraction mapping theorem is also proved.

Keywords : Dislocated fuzzy metric space, dislocated neighbourhood, d-convergent.

1. Introduction:

Zadeh's introduction[15] of the notion of fuzzy sets laid down the foundation of fuzzy mathematics. In the last two decades there were a tremendous growth in fuzzy mathematics. Many fixed point theorems for contractions in fuzzy metric spaces appeared (see [1],[2],[4],[5],[8-10],[12-14]). The role of topology in logic programming has come to be recognized in recent years. In particular topological methods are employed in order to obtain fixed point semantics for logic programs. Motivated by this fact Hitzler and Seda [6] introduced the concept of dislocated metric space and studied dislocated topologies associated with it. They also proved a generalized version of Banach contraction mapping theorem which was applied to obtain fixed point semantics for logic programs. In this paper we have introduced the concept of dislocated fuzzy metric spaces and studied the fuzzy topology associated with it. We have also proved two fixed point theorems in DFM-Space, which extends and generalizes the results of Grabiec [3], Gregory and Romuguera [4], Gregory and Sapena [5], Mihet [8] and Radu [10].

2. Preliminaries

Definition 2.1 [11] : A binary operation $*$: [0,1] x [0,1] $\to$ [0,1] is a continuous t-norm if ([0,1], $*$) is an abelian monoid with unit one such that, for all a,b,c,d in [0,1], $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Definition 2.2 [6]: Let X be a set. A relation $\succ \subseteq X \times P(X)$ is called a *d-membership* relation on *X* if it satisfies the following property:

$x \succ A$ and $A \subset B$ implies $x \succ B$ $\forall$ $x \in X$, and $A,B \in P(X)$, where *P(X)* is the power set of *X* .

If $x \succ A$ we read it as *x* is below *A*.

Definition 2.3 [6]: Let *X* be a set, $\succ$ be a d-membership relation on *X*. For each $x \in X$, let U_x be the collection of all subsets of *X* satisfying the following conditions:
(N_1) if $U \in U_x$ then $x \succ U$.
(N_2) if $U,V \in U_x$ then $U \cap V \in U_x$.
(N_3) if $U \in U_x$ then there is a $V \subset U$ with $V \in U_x$ such that for all $y \succ V$ we have $U \in U_y$.
(N_4) if $U \in U_x$ and $U \subset V$ then $V \in U_x$.

Then $(U_x, \succ)$ is called a *d-neighborhood system* for *x* and each $U \in U_x$ is called a d-neighborhood of *x*.

If $U = \{U_x : x \in X\}$ then $(X, U \succ)$ is called a *dislocated topological space* or a *d-topological space*.

Definition 2.4 [6]: Let $(X, U, \succ)$ be a d-topological space and $x \in X$. A topological net $\langle x_\lambda \rangle$ *d-converges* to $x \in X$, if for each d-neighborhood U of *x*, we have x_λ is eventually in U. i.e. there exist λ_0, such that $x_\lambda \in U$ for all $\lambda > \lambda_0$.

3. Dislocated Fuzzy Metric Space and Topologies

In this section we introduce the concept of dislocated fuzzy metric space and discuss the fuzzy topologies associated with it. We also prove a generalized fuzzy version of Banach contraction mapping theorem.

Definition 3.1: Let *X* be any non empty set, $*$ be a continuous t-norm and $M : X^2 \times [0,\infty) \to [0,1]$ be a fuzzy set. Consider the following conditions:

For all $x, y, z \in X$ and $t, s \in [0,\infty)$

$FM1 \quad M(x,y,0) = 0$

$FM2 \quad M(x,x,t) = 1$

$FM3 \quad M(x,y,t) = 1 \Rightarrow x = y$

$FM4 \quad M(x,y,t) = M(y,x,t)$

$FM5 \quad M(x,y,t+s) \geq M(x,z,t) * M(z,y,s)$

$FM6 \quad M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous

If *M* satisfies conditions *FM*1 to *FM*6 then $(X,M, *)$ is called a Fuzzy Metric Space [7].

If *M* satisfies conditions *FM*1 and *FM*3 to *FM*6 then we say that $(X,M, *)$ is a *Dislocated Fuzzy Metric Space* (in short DFM – Space).

Example 3.1 : Let $X = R$; Define $a * b = ab$,

$$M(x,y,t) = \left[\exp\left(\frac{|x-y|+|x|+|y|}{t}\right)\right]^{-1}$$

for all $(x,y) \in X \times X$, $t \in [0,\infty)$, $(x,y) \neq (0,0)$ and $M(0,0,0) = 0$. Then $(X,M, *)$ is a DFM-Space.

Proof: ***FM*1** Clearly *M(x, y, 0) = 0*

***FM*3** $M(x,y,t) = 1$

$\Rightarrow |x-y|+|x|+|y| = 0 \Rightarrow x = 0 = y$.

***FM*4** Clearly *M(x,y,t) = M(y,x,t)*

***FM*5** We have

$$|x-z|+|x|+|z| \leq \frac{t+s}{t}\left[|x-y|+|x|+|y|\right] + \frac{t+s}{s}\left[|y-z|+|y|+|z|\right]$$

i.e. $\frac{|x-z|+|x|+|z|}{t+s} \leq \frac{|x-y|+|x|+|y|}{t} + \frac{|y-z|+|y|+|z|}{s}$

$$\therefore \left[\exp\left(\frac{|x-z|+|x|+|z|}{t+s}\right)\right]^{-1} \geq \left[\exp\left(\frac{|x-y|+|x|+|y|}{t}\right)\right]^{-1}\left[\exp\left(\frac{|y-z|+|y|+|z|}{s}\right)\right]^{-1}$$

$$\therefore M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$$

***FM*6** Clearly $M(x,y, .) : [o, \infty) \to [0,1]$ is continuous

Now we give some definitions needed for stating the generalized fuzzy version of Banach Contraction Mapping Theorem.

Definition 3.2: A sequence $\langle x_n \rangle$ in a DFM-Space $(X,M, *)$ is said to be *d-convergent* if $\exists\, x \in X$ such that $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t > 0. We say that *x* is the limit of $\langle x_n \rangle$ and we write $lim_n\, x_n = x$.

Proposition 3.1 : Limits in DFM-Spaces are unique.

Proof: Let $\langle x_n \rangle$ be a sequence in $(X,M, *)$, *x* and *y* be its limits. By *FM*5 we have $M(x,y,t) \geq M(x, x_n, \frac{t}{2}) * M(x_n, y, \frac{t}{2})$. Taking the limits as $n \to \infty$ we have $M(x,y,t) \geq 1 * 1 = 1$. Hence *x = y*.

Definition 3.3 : A sequence $\langle x_n \rangle$ in a DFM-Space $(X,M, *)$ is said to be a *Cauchy sequence* if for each *t* > 0, and $p \in N$, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$.

Definition 3.4 : A DFM-Space $(X,M, *)$ is said to be *complete* if every Cauchy sequence in it is d-convergent.

Now we investigate some topological points of view of dislocated fuzzy metrics.

Definition 3.5 : Let $(X,M, *)$ be a DFM-Space.We define an *open ball* with centre *x* and radius *r*, $(0 < r < 1)$ in *X* as $B(x,r) = \{y \in X : M(x,y,t) > 1 - r$ for all $t \in [0,\infty)\}$

Proposition 3.2: Let $(X,M, *)$ be a DFM-Space. Define a d-membership relation $\succ$ by $x \succ A$ if and only if there exists *r* $(0 < r < 1)$ such that $B(x,r) \subseteq A$. For each $x \in X$, let U_x be the collection of subsets *A* of *X* such that $x \succ A$. Then $(U_x, \succ)$ is a d-neighborhood system for *x*.

Proof: **(N_1)** Clearly if $U \in U_x$ then $x \succ U$ and vice versa.

(N_2) Let $U,V \in U_x$, i.e. $x \succ U$ and $x \succ V$. Then there exists $0 < r_1 < 1$ and $0 < r_2 < 1$ such that $B(x,r_1) \subseteq U$ and $B(x,r_2) \subseteq V$. Suppose $r_1 < r_2$. Then we have $1-r_1 > 1-r_2$. Let $y \in B(x,r_1)$ .Then $M(x,y,t) > 1-r_1 \Rightarrow M(x,y,t) > 1-r_2 \Rightarrow y \in B(x,r_2)$ Thus $B(x,r_1) \subseteq B(x,r_2)$. Hence $B(x,r_1) = B(x,r_1) \cap B(x,r_2) \subseteq U \cap V \Rightarrow x \succ U \cap V$.

(N_3) Let $U \in U_x$ i.e. $x \succ U$. Then, there exists $0 < r < 1$ such that $B(x,r) \subseteq U$. Also $B(x,r) \in U_x$. Let $y \succ B(x,r)$ be arbitrary. Then there exist $0 < r_1 < 1$ such that $B(y,r_1) \subseteq B(x,r)$ .But $y \succ B(y,r_1)$. Thus we have $y \succ B(y,r_1) \subseteq B(x,r) \subseteq U \Rightarrow y \succ U$, i.e. $U \in U_y$.

(N_4) This is obvious.

Remark 1: The above construction yields the usual dislocated fuzzy topology associated with a dislocated fuzzy metric.

Proposition 3.3 : Let $(X, M, *)$ be a DFM-Space and let $(X,U,\succ)$ be the D-Fuzzy topological space obtained from it via the construction in the proof of Proposition 3.2. Let $\langle x_n \rangle$ be a sequence in X. Then $\langle x_n \rangle$ d-converges in $(X, M, *)$ if and only if $\langle x_n \rangle$ d-converges in $(X,U,\succ)$.

Proof : Let $\langle x_n \rangle$ be d-convergent in $(X, M, *)$ to some $x \in X$, i.e. $M(x_n, x, t) \to 1$ as $n \to \infty$. Let U be a d-neighbourhood of x, i.e. there exist $0 < r < 1$ such that $B(x,r) \subseteq U$. Since $M(x_n, x, t) \to 1$ as $n \to \infty$, we see that there exist $n_0 \in N$, such that $M(x_n, x, t) > 1-r \ \forall\, n \ge n_0$ $\Rightarrow x_n \in B(x,r) \subseteq U \ \forall\, n \ge n_0$. Therefore $\langle x_n \rangle$ d-converges in $(X,U,\succ)$. Conversely, let $\langle x_n \rangle$ be d-convergent in $(X,U,\succ)$ to some $x \in X$, i.e. for each d-neighbourhood U of x, there exist $n_0 \in N$ such that $x_n \in U \ \forall\, n \ge n_0$. For each $r > 0$, $B(x,r)$ is a d-neighbourhood of x. $\therefore x_n \in B(x,r)$ for all $n \ge n_0$, i.e. $M(x_n, x, t) > 1-r \ \forall n \ge n_0$ and $t \in [0,\infty)$ $\Rightarrow M(x_n, x, t) \to 1$ as $n \to \infty$. Hence $\langle x_n \rangle$ d-converges to x in $(X, M, *)$.

Definition 3.6 : Let $(X, M, *)$ be a dislocated fuzzy metric space, f and g be self maps of X. f and g are said to commute at $x \in X$, if and only if, $M(fgx, gfx, t) = 1$ for all $t \in [0,\infty)$. If f and g commute at all $x \in X$, then we say that f and g are commuting on X.

Definition 3.7 : Mappings f and g are said to be coincidentally commuting if and only if they commute at all the coincidence points of f and g.

There exists mappings f and g which commutes at some coincidence points but do not commute at all coincidence points of f and g.

Theorem 3.1 : Let $(X,M, *)$ be a DFM-Space and let $f, g : X \to X$ be mappings that satisfy the following conditions:

3.1.1) $f(X) \subset g(X)$
3.1.2) one of $f(X)$ or $g(X)$ is complete.
3.1.3) $M(fx, fy, kt) \ge M(gx, gy, t)$ for all $x, y \in X$, $0 < k < 1$, $t \in [0,\infty)$.
3.1.4) $\lim_{t\to\infty} M(x,y,t) = 1$

Then f and g have a coincidence point.
Further if f and g commute at some coincidence point, then f and g have a unique common fixed point.

Proof: For some $x_0 \in X$ by (3.1.1) we can find $x_1 \in X$ such that $y_1 = fx_0 = gx_1$.

By induction we can form the sequence $\langle y_n \rangle$ such that $y_{n+1} = fx_n = gx_{n+1}$, n = 0,1,2…… and $y_0 = gx_0$. For $0 < k < 1$ and $t \in [0,\infty)$ we have

$M(y_1, y_2, kt) = M(fx_0, fx_1, kt) \ge M(gx_0, gx_1, t) = M(y_0, y_1, t)$
$M(y_2, y_3, kt) = M(fx_1, fx_2, kt) \ge M(gx_1, gx_2, t) = M(y_1, y_2, t)$
$\ge M(y_0, y_1, \frac{t}{k})$. $\therefore M(y_2, y_3, t) \ge M(y_0, y_1, \frac{t}{k^2})$.

Proceeding this way, we have $M(y_n, y_{n+1}, t) \ge M(y_0, y_1, \frac{t}{k^n})$. $\therefore$ For $p \in N$, $t \in [0,\infty)$ we have

$M(y_n, y_{n+p}, t) \ge M(y_n, y_{n+1}, \frac{t}{2}) * M(y_{n+1}, y_{n+p}, \frac{t}{2})$
$\ge M(y_n, y_{n+1}, \frac{t}{2}) * M(y_{n+1}, y_{n+2}, \frac{t}{4}) * M(y_{n+2}, y_{n+p}, \frac{t}{4})$
$\ge M(y_n, y_{n+1}, \frac{t}{2}) * M(y_{n+1}, y_{n+2}, \frac{t}{4}) * \ldots * M(y_{n+p-1}, y_{n+p}, \frac{t}{2^p})$

$\geq M(y_0, y_1, \frac{t}{2k^n}) * M(y_0, y_1, \frac{t}{4k^{n+1}}) * ... * M(y_0, y_1, \frac{t}{2^p k^{n+p-1}})$

Taking the limit as $n \to \infty$ we get $M(y_n, y_{n+p}, t) \geq 1 * 1 * ... * 1$,

i.e. $\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1$. Hence $\langle y_n \rangle$ is a Cauchy sequence. Now suppose *g(X)* is complete. Then there exists $u \in$ *g(X)* such that $\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} gx_{n+1} = u = \lim_{n\to\infty} fx_n$. Let $gp = u$ for some $p \in X$. Then

$M(fp, gp, kt) = \lim_{n\to\infty} M(fp, fx_n, kt)$

$\geq \lim_{n\to\infty} M(gp, gx_n, t) = \lim_{n\to\infty} M(u, gx_n, t) = 1.$

Therefore $fp = gp$ i.e. *f* and *g* has a coincidence point. Now suppose *f* and *g* commute at some coincidence point say z, i.e. $M(fgz, gfz, t) = 1 \quad \forall t > 0$ $\Rightarrow \quad fgz = gfz \quad [by FM3]$. Let $fz = gz = v$. Then we have

$M(fv, v, kt) \geq M(fv, fz, kt) \geq M(gv, gz, t) = M(gfz, fz, t)$

$= M(fgz, fz, t) \geq M(gv, gz, \frac{t}{k}) \geq ... \geq M(gv, gz, \frac{t}{k^n}),$

as $n \to \infty$ we get $fv = v$. Similarly we can show that $gv = v$. That is *v* is a common fixed point of *f* and *g*. A similar proof follows if *f(X)* is complete. Now suppose *w* is another common fixed point of *f* and *g*. Then we have

$M(v, w, kt) = M(fv, fw, kt) \geq M(gv, gw, t)$

$= M(v, w, t) = M(fv, fw, t) \geq M(gv, gw, \frac{t}{k})$

$= M(v, w, \frac{t}{k}) = M(fv, fw, \frac{t}{k}) \geq M(gv, gw, \frac{t}{k^2})$

$\ldots\ldots \geq M(v, w, \frac{t}{k^n})$. As $n \to \infty$ we get $M(v, w, kt) \geq 1$. Hence *v* = *w*.

In Theorem 3.1 , if we take $g = I_X$, the identity mapping on *X*, then we have the following.

Theorem 3.2 : Let (*X*,*M*, $*$) be a DFM-Space and let $f : X \to X$ be mappings that satisfy the following conditions:

3.1.5) *f*(X) is complete.

3.1.6) $M(fx, fy, kt) \geq M(x, y, t)$ for all $x, y \in X, \quad 0 < k < 1, \quad t \in [0, \infty)$.

3.1.7) $\lim_{t\to\infty} M(x, y, t) = 1$

Then *f* has a unique fixed point.

Remark 3 : Theorem 3.2 is a generalized fuzzy version of Banach contraction mapping theorem.

In recent years various generalizations of Banach contraction theorem appeared in Fuzzy metric spaces. (See [4],[5],[8] and [10]). In the next result we prove a fixed point theorem in DFM-Space which extends and generalizes the results of [4],[5],[8] and [10].

Theorem 3.3 : Let (*X*,*M*, $*$) be a DFM-Space and let $f, g : X \to X$ be mappings that satisfy the following conditions:

3.3.1) $f(X) \subset g(X)$

3.3.2) one of *f*(*X*) or *g*(*X*) is complete.

3.3.3) $x, y \in X, t > 0, \ M(gx, gy, t) > 0$
$\Rightarrow M(fx, fy, t) \geq \psi(M(gx, gy, t))$

where $\psi : (0,1] \to (0,1]$ is increasing and $\lim_{n\to\infty} \psi^n(\lambda) = 1$ and $\psi(\lambda) \geq \lambda$ for all $\lambda \in (0,1]$.

3.3.4) $M(g(x), f(x), t) > 0 \ \forall t > 0$, for some $x \in X$

Then *f* and *g* have a coincidence point. Further if *f* and *g* are coincidentally commuting and $M(ggz, gz, t) > 0$ for some $z \in C(f, g)$ { $C(f, g)$ denotes the set of coincidence points of *f* and *g*}and all *t > 0* , then *f* and *g* have a common fixed point.

Proof. By (3.3.4), there exist some $x_0 \in X$, such that $M(gx_0, fx_0, t) > 0 \ \forall t > 0$. Proceeding as in Theorem 3.1 we can construct sequence $\{y_n\}$ such that $y_{n+1} = fx_n = gx_{n+1}$, n = 0,1,2…… and $y_0 = gx_0$. Now since $M(gx_0, fx_0, t) > 0$, by (3.3.3) we get $M(y_{n+1}, y_{n+2}, t) \geq \psi^n(M(gx_0, fx_0, t))$ for all $n \in N$ and $t > 0$, and hence as $n \to \infty$, $M(y_{n+1}, y_{n+2}, t) \to 1$ for t > 0. . Thus for $m \in N$, $M(x_n, x_{n+m}, t) \geq M(x_n, x_{n+1}, \frac{t}{m}) * M(x_{n+1}, x_{n+2}, \frac{t}{m})$ $* ... * M(x_{n+m-1}, x_{n+m}, \frac{t}{m}) \to 1$. Hence sequence $\{y_n\}$ is a Cauchy sequence. Now suppose *g(X)* is complete. Then there exists $u \in$ *g(X)* such that $\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} gx_{n+1} = u = \lim_{n\to\infty} fx_n$. Let $gp = u$ for some $p \in X$. Then by definition $M(gp, fx_n, t) \to 1 \Rightarrow M(gp, gx_{n+1}, t) \to 1$. By (3.3.3) $M(fp, fx_{n+1}, t) \geq \psi(M(gp, gx_{n+1}, t))$ $\geq M(gp, gx_{n+1}, t) \to 1$. Thus we have $M(fp, gp, t) = \lim_{n\to\infty} M(fp, fx_{n+1}, t) \to 1$. Hence $fp = gp$. Now let *f* and *g* be coincidentally commuting and $M(ggz, gz, t) > 0$ for some $z \in C(f, g)$ and all $t > 0$. Let $gz = fz = v$. By (3.3.3) we have $M(fgz, fz, t) \geq \psi(M(ggz, gz, t) > 0$

$\Rightarrow M(gfz, gz, t) > 0$

$\Rightarrow M(ffz, fz, t) \geq \psi(M(gfz, gz, t))$

……

$\Rightarrow M(ffz, fz, t) \geq \psi^{n}(M(gfz, gz, t)) \to 1$. Hence $ffz = fz$, i.e. $fv = v$. Also $gfz = fgz = ffz = fz$, i.e. $gv = v$. Thus v is a common fixed point of f and g. A similar proof follows if $f(X)$ is complete.

4. Conclusions

In this work we have studied the concept of dislocated fuzzy metrics and underlying dislocated fuzzy topology. We have proved two fixed point theorems in dislocated fuzzy metric space which extends and generalizes many known results, particularly the Banach contraction mapping theorem. Although a few applications of dislocated metrics and in particular the Banach contraction mapping theorem in dislocated metric spaces is known in theoretical computer science (logic programming semantics), at this stage it is worth investigating, whether or not some applications of dislocated fuzzy metrics can be found, particularly in relationship to the semantics of probabilistic logic programs or fuzzy logic programs .

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