Algorithms for Operations on Pinned-Flags

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Abstract

Fuzzy subgroups of finite abelian groups have been treated recently using the concept of pinned-flags. In this paper we develop algorithms to construct pinned-flags corresponding to operations of intersection, sum, product and quotient on fuzzy subgroups.

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1 Introduction.

A pinned-flag on a finite group of order N is a pair consisting of a flag (a maximal chain of subgroups) and a keychain (a $(n+1)$-tuple of real numbers starting with 1 in the descending order from the unit interval not necessarily all distinct). One can study the operations on fuzzy subgroups by means of pinned-flags, thus enriching some properties of fuzzy subgroups. It was observed in [2] that the pinned-flag resulting from operations of intersection and direct sum of fuzzy subgroups do not form any particular pattern. We develop some interesting algorithms here to describe the pinned-flags of the intersection, sum, product and quotient of fuzzy subgroups. We use $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$, the real unit interval as a chain with the usual ordering in which $\wedge$ stands for infimum (inf) (or intersection) and $\vee$ stands for supremum (sup)(or union).

2 Preliminaries.

A *fuzzy subset* of a set G is a mapping $\mu : G \to \mathbf{I}$. We denote fuzzy subsets by the Greek letters μ, ν, η, etc. Throughout this paper we take G to be a finite abelian group of order N and G_0 to be the trivial subgroup $\{0\}$. Even though almost all results of this paper are applicable to any finite group, abelian or not , we use $+$ for the group operation and 0 for the identity element for convenience of notation, rather than the more conventional multiplicative notation. By an α-cut of μ, for a real number α in $\mathbf{I}$, we mean a subset $\mu^\alpha = \{x \in G : \mu(x) \geq \alpha\}$ of G. Without loss of generality, we assume $\mu(0) = 1$. This assumption implies that the only admissible fuzzy subgroup of the trivial group is $\mu(0) = 1$. By *core* and *support* of μ we mean the crisp subsets of G given by $\{x \in G : \mu(x) = 1\}$ and $\{x \in G : \mu(x) \neq 0\}$ respectively. For results on product and sum [4].

2.1 Flags and Keychains

We refer the reader to [2] for results on flag, keychain and pinned-flag, but state their definitions here. By a *flag* $\mathcal{C}$ on G we mean a maximal chain of subgroups of the form

$$\{0\} = G_0 \subset G_1 \subset G_2 \cdots \subset G_n = G \quad (2.1)$$

We call various G_i's the *components* of the flag $\mathcal{C}$; In particular G_i is called the ith component of the flag. From the Jordan-Holder Theorem it is clear that any two flags of G are of the same length. We assume this length to be $n+1$ for some fixed n less than N. By a *keychain* ℓ we mean an (n+1)-tuple $(\lambda_0, \lambda_1, \cdots, \lambda_n)$ of real numbers in $\mathbf{I}$, called *pins*, of the form

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \quad (2.2)$$

Definition 2.1 *By a pinned-flag on G, we mean a pair ($\mathcal{C}, \ell$), of a flag $\mathcal{C}$ on G and a keychain ℓ from* $\mathbf{I}$*, written as follows:*

$$G_0^1 \subset G_1^{\lambda_1} \subset G_2^{\lambda_2} \cdots \subset G_n^{\lambda_n} \quad (2.3)$$

We call $G_i^{\lambda_i}$ for $i = 0, 1, \cdots, n$, the i-th component of the pinned-flag.
With the pinned-flag ($\mathcal{C}, \ell$) we can associate a fuzzy subgroup μ as follows:

$$\mu(x) = \begin{cases} 1, & x = 0 \\ \lambda_1, & x \in G_1 \setminus \{0\} \\ \lambda_2, & x \in G_2 \setminus G_1 \\ \vdots & \\ \lambda_n, & x \in G_n \setminus G_{n-1} \end{cases} \quad (2.4)$$

It is easily checked that μ as defined above is fuzzy subgroup of G.
Conversely, one can associate a pinned-flag $(\mathcal{C}_\mu, \ell_\mu)$ with a given fuzzy subgroup μ of G.

3 Algorithms for Pinned-flags

In this section we study the operations of intersection, sum (product is similar to sum) and quotient of fuzzy subgroups through their pinned-flags.

3.1 Algorithm for Intersection

Suppose the pinned-flags corresponding to two fuzzy subgroups μ and ν of G are given by

$$\begin{array}{c} (\mathcal{C}_\mu, \updownarrow_\mu) : G_0^1 \subset G_1^{\lambda_1} \subset \cdots \subset G_n^{\lambda_n} \\ \text{and} \\ (\mathcal{C}_\nu, \updownarrow_\nu) : H_0^1 \subset H_1^{\beta_1} \subset \cdots \subset H_n^{\beta_n} \end{array} \quad (3.1)$$

The pinned-flag $K_0^1 \subset K_1^{\gamma_1} \subset \cdots \subset K_n^{\gamma_n}$ for $\mu \wedge \nu$ is constructed as follows:

Algorithm 3.1 *Step 1: Firstly* $K_0^1 = G_0^1 = H_0^1$. *We usually denote this by simply* 0^1 *in all cases.*
Step 2:
To find $K_1^{\gamma_1}$, *the second component of the pinned-flag, we proceed as follows: If* $H_1 \cap G_1 \neq K_0$, *then* $G_1 = H_1$ *by maximality of the chains, thus* $K_1 = G_1 = H_1$ *and* $\gamma_1 = \lambda_1 \wedge \beta_1$.
If $H_1 \cap G_1 = K_0$, *then* $H_1 \neq G_1$. *Now find the least i and j such that* $G_1 \cap H_i \neq K_0$ *for* $1 < i \leq n$ *and* $H_1 \cap G_j \neq K_0$ *for* $1 < j \leq n$ *respectively. Then* $\gamma_1 = (\lambda_1 \wedge \beta_i) \vee (\lambda_j \wedge \beta_1)$ *and*

$$K_1 = \begin{cases} G_1 \cap H_i = G_1 & \text{if} \quad \gamma_1 = \lambda_1 \wedge \beta_i \\ G_j \cap H_1 = H_1 & \text{if} \quad \gamma_1 = \lambda_j \wedge \beta_1 \end{cases} \quad (3.2)$$

Step 3:
To compute $K_2^{\gamma_2}$ *we proceed as follows:*
If $H_2 \cap G_2 \neq K_1$, *then* $G_2 = H_2$ *by maximality of the chains, thus* $K_2 = G_2 = H_2$ *and* $\gamma_2 = \lambda_2 \wedge \beta_2$. *If* $H_2 \cap G_2 = K_1$, *then* $H_2 \neq G_2$. *In this case find the least i and j such that* $G_2 \cap H_i \neq K_1$ *for* $2 < i \leq n$ *and* $H_2 \cap G_j \neq K_1$ *for* $2 < j \leq n$ *respectively. Then* $\gamma_2 = (\lambda_2 \wedge \beta_i) \vee (\lambda_j \wedge \beta_2)$ *and*

$$K_2 = \begin{cases} G_2 \cap H_i = G_2 & \text{if} \quad \gamma_2 = \lambda_2 \wedge \beta_i \\ G_j \cap H_2 = H_2 & \text{if} \quad \gamma_2 = \lambda_j \wedge \beta_2 \end{cases} \quad (3.3)$$

Step 4: (Inductive step).
Suppose we have defined $K_s^{\gamma_s}$. *To obtain* $K_{s+1}^{\gamma_{s+1}}$ *we proceed as follows:*
If $H_{s+1} \cap G_{s+1} \neq K_s$, *then* $G_{s+1} = H_{s+1}$ *by maximality of the chains, thus* $K_{s+1} = G_{s+1} = H_{s+1}$ *and* $\gamma_{s+1} = \lambda_{s+1} \wedge \beta_{s+1}$. *If* $H_{s+1} \cap G_{s+1} = K_s$, *then* $H_{s+1} \neq G_{s+1}$. *In this case find the least* i *and* j *such that* $G_{s+1} \cap H_i \neq K_s$ *for* $s+1 < i \leq n$ *and* $H_{s+1} \cap G_j \neq K_s$ *for* $s+1 < j \leq n$ *respectively. Then* $\gamma_{s+1} = (\lambda_{s+1} \wedge \beta_i) \vee (\lambda_j \wedge \beta_{s+1})$ *and*

$$K_{s+1} = \begin{cases} G_{s+1} \cap H_i = G_{s+1} \\ G_j \cap H_{s+1} = H_{s+1} \end{cases} \tag{3.4}$$

if $\gamma_{s+1} = \lambda_{s+1} \wedge \beta_i$ *in the first case and if* $\gamma_{s+1} = \lambda_j \wedge \beta_{s+1}$ *in the second case. If* $K_s = K_n$, *the algorithm terminates.*

We wish to point out in the above algorithm that the determination of K_i's is dependent on the values of γ_i's, which in turn depend on the λ_i's and β_j's.

3.2 Algorithm for sum

We next construct the pinned-flag of the sum $\mu + \nu$ of two fuzzy subgroups from their associated pinned-flags. Suppose μ and ν are represented by pinned-flags as in equation 3.1. Algorithm to construct the pinned-flag $K_0^1 \subset K_1^{\gamma_1} \subset \cdots \subset K_n^{\gamma_n}$ for $\mu + \nu$ is as follows:

Algorithm 3.2 *Step 1: Firstly* $K_0^1 = G_0^1 = H_0^1$. *We usually denote this by simply* 0^1 *in all cases.*
Step 2:
To find $K_1^{\gamma_1}$, *the second component of the pinned-flag, we proceed as follows: If* $G_1 = H_1$ *then* $K_1^{\gamma_1} = G_1^{\lambda_1 \vee \beta_1} = H_1^{\lambda_1 \vee \beta_1}$.
If $G_1 \neq H_1$, *then*

$$K_1^{\gamma_1} = \begin{cases} G_1^{\lambda_1} & \text{if} \quad \lambda_1 \geq \beta_1 \\ H_1^{\beta_1} & \text{if} \quad \beta_1 \geq \lambda_1 \end{cases} \tag{3.5}$$

Step 3: (Inductive step).
Suppose we have defined $K_s^{\gamma_s}$. *To obtain* $K_{s+1}^{\gamma_{s+1}}$ *we proceed as follows:*
If $G_{s+1} = H_{s+1}$ *then find the least* i *and* j, *if they exist, such that* $G_{s+1} = G_i + H_j$ *for* $i < s+1$ *and* $j < s+1$. *In this case*

$$K_{s+1}^{\gamma_{s+1}} = G_{s+1}^{\lambda_i \wedge \beta_j} = H_{s+1}^{\lambda_i \wedge \beta_j}. \tag{3.6}$$

If $G_{s+1} = H_{s+1}$ *but* i *and* j *as cited above do not exist, then*

$$K_{s+1}^{\gamma_{s+1}} = \begin{cases} G_{s+1}^{\lambda_{s+1}} & \text{if} \quad \lambda_{s+1} \geq \beta_{s+1} \\ H_{s+1}^{\beta_{s+1}} & \text{if} \quad \beta_{s+1} \geq \lambda_{s+1} \end{cases} \tag{3.7}$$

If $G_{s+1} \neq H_{s+1}$, *but* $G_{s+1} = G_i + H_j$ *for some least* i *and* j *and* $H_{s+1} \neq G'_i + H'_j$ *for any* i' *and* j' *then*

$$K_{s+1}^{\gamma_{s+1}} = \begin{cases} G_{s+1}^{\lambda_i \wedge \beta_j} & \text{if} \quad \lambda_i \wedge \beta_j \geq \beta_{s+1} \\ H_{s+1}^{\beta_{s+1}} & \text{if} \quad \beta_{s+1} \geq \lambda_i \wedge \beta_j \end{cases} \tag{3.8}$$

Similar formula if H_{s+1} *splits but* G_{s+1} *does not split. When both of them split such as* $G_{s+1} = G_i + H_j$ *and* $H_{s+1} = G'_i + H'_j$ *then*

$$K_{s+1}^{\gamma_{s+1}} = \begin{cases} G_{s+1}^{\lambda_i \wedge \beta_j} & \text{if} \quad \lambda_i \wedge \beta_j \geq \lambda'_i \wedge \beta'_j \\ H_{s+1}^{\lambda'_i \wedge \beta'_j} & \text{if} \quad \lambda'_i \wedge \beta'_j \geq \lambda_i \wedge \beta_j \end{cases} \tag{3.9}$$

When neither G_{s+1} *nor* H_{s+1} *splits, then*

$$K_{s+1}^{\gamma_{s+1}} = \begin{cases} G_{s+1}^{\lambda_{s+1}} & \text{if} \quad \lambda_{s+1} \geq \beta_{s+1} \\ H_{s+1}^{\beta_{s+1}} & \text{if} \quad \beta_{s+1} \geq \lambda_{s+1} \end{cases} \tag{3.10}$$

If $K_s = K_n$, *the algorithm terminates.*

3.3 Pinned-flags for quotient

Suppose μ and ν are two fuzzy subgroups of G. Then the *fuzzy quotient group*

μ/ν is defined as a fuzzy subgroup of the quotient group $G/core(\nu)$ given by $(\mu/\nu)(x\, core(\nu)) = \sup\{\mu(a) : a\, core(\nu) = x\, core(\nu),\ a \in G\}$, [1].

Algorithm 3.3 *Suppose the pinned-flags for μ and ν are given as in equation 3.1. Since μ and ν are fuzzy subgroups of the same group G, $m = n$. If the $core(\nu) = G$ then the quotient is $G/core(\nu) = \{e\}$. In this case the pinned-flag of the quotient is simply $(G/core(\nu))^1$ with only one component.*
Assume that $core(\nu) \neq G$. So the $core(\nu) = H_i$ for some $0 \leq i < n$.
Case(i): Let $H_i = G_j$. Then firstly $i = j$. If not, either $i > j$ or $i < j$. The case $i < j$ implies that the maximal chain

$$G_0 \subset G_1 \subset \cdots \subset G_{j-1} \subset G_j = H_i \subset H_{i+1} \subset \cdots \subset H_n \tag{3.11}$$

has length $n-i+1+j = n+1+(j-i) > n+1$, a contradiction. Similarly the case $i > j$ leads to a contradiction. Secondly

$$H_i/H_i = G_i/H_i \subset G_{i+1}/H_i \subset \cdots \subset G_n/H_i$$

is clearly a maximal chain in G/H_i, leading to a pinned-flag

$$(G_i/H_i)^1 \subset (G_{i+1}/H_i)^{\lambda_{i+1}} \subset \cdots \subset (G_n/H_i)^{\lambda_n}$$

for the quotient μ/ν.
Case (ii): $H_i \neq G_i$. Choose the least subscript k such that $H_i \subset G_k$. Clearly the weighted chain

$$(G_{k+1}/H_i)^{\lambda_{k+1}} \subset \cdots \subset (G_n/H_i)^{\lambda_n}$$

is part of the pinned-flag for μ/ν. We now extend the above weighted chain to a full pinned-flag for μ/ν. Choose the least subscript $l_1 \leq k$ such that $G_{l_1}H_i = G_k$. Inductively continue to find the least l_i such that $k \geq l_1 > l_2 > \cdots > l_s$ and $G_{l_2}H_i = G_{l_1-1}H_i, \cdots, G_{l_s}H_i = H_i$. Consequently the pinned-flag for the quotient μ/ν is given by

$$(H_i/H_i)^1 \subset (G_{l_{s-1}}H_i/H_i)^{\lambda_{l_{s-1}}} \subset \cdots \subset (G_{l_1}H_i/H_i)^{\lambda_{l_1}} \subset (G_{k+1}/H_i)^{\lambda_{k+1}} \subset (G_{k+2}/H_i)^{\lambda_{k+2}} \subset \cdots \subset (G_n/H_i)^{\lambda_n} \tag{3.12}$$

In conclusion, we wish to emphasize that the study of operations on fuzzy subgroups through their pinned-flags is important. The proofs of validity of the above algorithms can be found in [3]. This paper illustrates the way the operations are tied up with pinned-flags of equivalent fuzzy subgroups.

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