On Idempotent Fuzzy Vector Spaces

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Abstract

The idempotency of fuzzy linear combinations of a finite set of idempotent matrices and that of their products are discussed. It is shown that the set of idempotent matrices with space ordering forms a fuzzy vector space. This leads to a factorization of Toeplitz matrices. The idempotency of the product of transitive closures of Fuzzy Retrival systems is discussed.

Keywords: Fuzzy vector space, space ordering, idempotent fuzzy matrix, Toeplitz matrix.

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1. Introduction

Throughout we deal with fuzzy matrices, that is, matrices over the fuzzy algebra $\mathcal{F} = [0, 1]$ under the max-min operations $(\oplus, \cdot)$ defined as $a \oplus b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for all $a, b \in \mathcal{F}$ and the standard order '$\leq$' of real numbers. The algebraic operations on matrices are max-min operations, which are different from that of the standard operations on real matrices. In practice, Fuzzy matrices have been proposed to represent Fuzzy relations in a system based on fuzzy sets theory [2], the behavior of the dynamic Fuzzy systems depends heavily on the products of Fuzzy matrices in the matrix representations of the system. Further, the powers of a Fuzzy matrix are either convergent to a Fuzzy matrix (or) oscillating with finite period. For a Fuzzy Matrix A, $A^{k+d} = A^k$ for some integers $k,d > 0$. Therefore, all Fuzzy matrices have an index and a period. On the other hand, most matrices over the nonnegative real numbers will not have an index and a period (section 4 [2]). Let $M_{m,n}(\mathcal{F})$ be the set of all $m \times n$ fuzzy matrices over $\mathcal{F}$. In short $M_{n,n}(\mathcal{F})$ is denoted as $M_n(\mathcal{F})$. $A \in M_{m,n}(\mathcal{F})$ is regular if there exists X such that $AXA = A$; X is called a generalized (g–) inverse of A and is denoted as A^-. $A\{1\}$, denotes the set of all g– inverses of a regular matrix A. $A \in M_n(\mathcal{F})$ is idempotent if $A^2 = A$. Clearly an idempotent matrix is a special case of regular matrices. $A \in M_n(\mathcal{F})$ is invertible if and only if A is a permutation matrix. Thus every invertible matrix is regular, where as the identity matrix is the only invertible matrix which is idempotent. In Fuzzy Retrival system, the degree of relevance of the concept matrix depends on that of its transitive closure, which is an idempotent matrix (pp 691-714,[1]). This motivates us to study on idempotent Fuzzy matrices.

It is known [2] that, every idempotent matrix can be expressed as a linear combination of idempotent Boolean matrices associated with it. On the other hand, a linear combination of idempotent fuzzy matrices need not be idempotent. In this paper, we investigate the idempotency of linear combinations of a finite set of idempotent matrices. In section 2, some basic definitions and results required are given. In section 3, a set of conditions for idempotency of linear combinations of a finite set of idempotent matrices are determined. By using this, we deduce that a fuzzy matrix is idempotent if and only if it is a linear combination of all its zero patterns, where each zero pattern is idempotent [2]. In section 4, conditions for the product of idempotent matrices to be idemponent are derived. In section 5, as an application, a factorization of Toeplitz matrices whose factors are all idempotent is derived. The problem of idempotency considered here asserts that the comparability of the concept matrices of any two Fuzzy Retrival systems is preserved for their corresponding transitive closures.

2. Preliminaries

For $A \in M_{mn}(\mathcal{F})$ the row space $\mathcal{R}(A)$ is a subspace of $M_{1n}(\mathcal{F})$ generated by the rows of A and the column space $\mathcal{C}(A)$ is defined in dual fashion. Let $M^-_{mn}(\mathcal{F})$ denotes the set of all regular matrices in $M_{mn}(\mathcal{F})$. Let $P_n(\mathcal{F})$ denotes the set of all idempotent fuzzy matrices in $M_n(\mathcal{F})$. We need the following result proved in [3].

Lemma 2.1.

For $A \in M_{mn}(\mathcal{F})$ and $B \in M^-_{mn}(\mathcal{F})$

$\mathcal{R}(A) \subseteq \mathcal{R}(B) \Leftrightarrow A = AB^-B$ for each $B^- \in B\{1\}$.

$\mathcal{C}(A) \subseteq \mathcal{C}(B) \Leftrightarrow A = BB^-A$ for each $B^- \in B\{1\}$.

We shall use the following terminologies. For a pair of matrices $A=(a_{ij})$ and $B=(b_{ij}) \in M_{mn}(\mathcal{F})$ B dominates A, denoted as $A \leq B$ if $a_{ij} \leq b_{ij}$ for all i, j. B space dominates A, denoted as $A \overset{s}{\leq} B$ if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{C}(A) \subseteq \mathcal{C}(B)$. A and B are comparable if either $A \le B$ (or) $B \le A$. A and B are space comparable if either $A \overset{s}{\le} B$ (or) $B \overset{s}{\le} A$.

Lemma 2.2.

For A, B $\in P_n(\mathcal{F})$ $A \overset{s}{\le} B \Leftrightarrow \mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{C}(A) \subseteq \mathcal{C}(B) \Leftrightarrow AB = A = BA$.

In our earlier work [4], we have proved that $M^{-}_{mn}(\mathcal{F})$ is a poset under minus ordering: $A \overset{-}{<} B$ if $A^{-}A = A^{-}B$ and $AA^{-} = BA^{-}$ for some $A^{-} \in A\{1\}$. From Lemma 2.2, we see that the minus ordering and space ordering are identical on $P_n(\mathcal{F})$. Thus $P_n(\mathcal{F})$ is a poset under space ordering: $A \overset{s}{\le} B$ for A, B $\in P_n(\mathcal{F})$.

3. Linear Combinations of Idempotent Matrices

In this section, we show that $P_n(\mathcal{F})$ is a vector space under the max-min operations. First, we derive a set of conditions for the idempotency of a linear combination of a finite set of idempotent matrices. Let N = {1, 2, . . . , k} be the index set.

Theorem 3.1.

Let $\{A_i / i \in N\}$ be a finite set of non-zero idempotent matrices in $P_n(\mathcal{F})$. Then $A = \overset{k}{\underset{i=1}{\oplus}} a_iA_i$ is idempotent for non-zero scalars $a_i \in \mathcal{F}$ if any one of the following conditions holds:

(i) $A_iA_j = A_jA_i = 0$.

(ii) $A_i^{'}s$ are pairwise comparable.

(iii) $A_i^{'}s$ are pairwise space comparable.

Proof. $A = \overset{k}{\underset{i=1}{\oplus}} a_iA_i$ is idempotent

$$\Leftrightarrow \overset{k}{\underset{i=1}{\oplus}} a_iA_i = \left(\overset{k}{\underset{i=1}{\oplus}} a_iA_i\right)\left(\overset{k}{\underset{i=1}{\oplus}} a_iA_i\right)$$

$$\Leftrightarrow \overset{k}{\underset{i=1}{\oplus}} a_iA_i = \overset{k}{\underset{i=1}{\oplus}} a_iA_i \overset{k}{\underset{\substack{i,j=1\\ i\ne j}}{\oplus}} a_ia_j A_iA_j \qquad (3.1)$$

Suppose (i) holds then (3.1) automatically holds.

Suppose (ii) holds, then for any $i \ne j \in N$, A_i and A_j are comparable. That is, either $A_i \le A_j$ (or) $A_j \le A_i$. Suppose $A_i \le A_j$, then $A_iA_j \le A_j$ and $A_jA_i \le A_j$. Further $a_ia_j \le a_j$, therefore, $a_ia_j A_iA_j \le a_jA_j$ and $a_ia_j A_jA_i \le a_jA_j$ By addition operation, this implies that

$$a_ia_jA_iA_j \oplus a_ia_j A_jA_i \oplus a_jA_j = a_jA_j \qquad (3.2)$$

Similarly if $A_j \le A_i$, since $a_ia_j \le a_i$ holds, we get

$$a_ia_jA_iA_j \oplus a_ia_j A_jA_i \oplus a_iA_i = a_iA_i \qquad (3.3)$$

Thus if A_i and A_j are comparable, then either (3.2) (or) (3.3) holds. Keep j fixed. Let $N^j = \{i / A_i \le A_j\}$ and $N_j = \{i / A_j \le A_i\}$. Clearly N^j and N_j are disjoint. Since A_j is comparable with all $A_i^{'}s$, equation (3.2) holds for all $i \in N^j$ and equation (3.3) holds for all $i \in N_j$. Now keeping j fixed and adding up all such (k-1) equations we get

$$\overset{k}{\underset{\substack{i=1\\ i\ne j}}{\oplus}} a_ia_jA_iA_j \overset{k}{\underset{\substack{i=1\\ i\ne j}}{\oplus}} a_ia_jA_jA_i \underset{i\in N_j}{\oplus} a_iA_i \oplus a_jA_j = \underset{i\in N_j}{\oplus} a_iA_i \oplus a_jA_j .$$

Now, allow j to vary over 1 to k, then we get

$$\overset{k}{\underset{\substack{i,j=1\\ i\ne j}}{\oplus}} a_ia_jA_iA_j \overset{k}{\underset{i=1}{\oplus}} a_iA_i = \overset{k}{\underset{i=1}{\oplus}} a_iA_i$$

Thus (3.1) holds and A is idempotent.

Suppose (iii) holds, then for $i \ne j \in N$, A_i and A_j are space comparable, that is, either $A_i \overset{s}{\le} A_j$ (or) $A_j \overset{s}{\le} A_i$. If $A_i \overset{s}{\le} A_j$, then by Lemma (2.2) $A_iA_j = A_i = A_jA_i$. Since $a_ia_j = a_ja_i \le a_i$, we get $a_ia_j A_iA_j \le a_iA_i$ and $a_ia_j A_jA_i \le a_iA_i$. By addition we get

$$a_ia_j A_iA_j \oplus a_ia_j A_jA_i \oplus a_iA_i = a_iA_i \qquad (3.4)$$

Similarly if $A_j \overset{s}{\le} A_i$, using $a_ia_j = a_ja_i \le a_j$, we get

$$a_ia_j A_iA_j \oplus a_ia_j A_jA_i \oplus a_jA_j = a_jA_j \qquad (3.5)$$

Thus if A_i and A_j are space comparable, then either (3.4) (or) (3.5) holds. By similar argument as in the previous case, we can show that (3.1) holds whenever $A_i^{'}s$ are pairwise space comparable. Hence A is idempotent. ■

Let $A \in M_{m,n}(\mathcal{F})$, and ϕ_A be the set of all non-zero entries of A. For $a \in \phi_A$, the zero pattern of A, denoted as A_a is a Boolean matrix defined by

$$[A_a]_{ij} = \begin{cases} 1 & \text{if } a_{ij} \ge a \\ 0 & \text{otherwise} \end{cases} .$$

It is well known [2] that $A \in M_{mn}(\mathcal{F})$ is expressed as a fuzzy linear combination of its associated Boolean matrices. Thus $A = \underset{a\in\phi_A}{\oplus} a A_a$.

For $a \ge b \in \phi_A$, we claim that $A_a \le A_b$ that is to prove $[A_a]_{ij} \le [A_b]_{ij}$ for all i, j. If $[A_a]_{ij} = 0 \le [A_b]_{ij}$; if $[A_a]_{ij} = 1$, then by definition $a_{ij} \ge a \ge b$ hence $[A_b]_{ij} = 1$. Thus in $\{A_a / a \in \phi_A\}$, the elements A_a's are comparable. By using this in Theorem 3.1, we deduce the result found in [2] in the following:

Corollary 3.2.
Let $A \in M_n(\mathcal{F})$; ϕ_A be the set of non-zero entries of A and $\{A_a / a \in \phi_A\}$ be the set of zero patterns of A. Then A = $\bigoplus_{a \in \phi_A} a A_a$ is idempotent $\Leftrightarrow$ each zero pattern A_a is idempotent.

In particular, the scalars $a_i = 1$ for each i $\in$ N, in Theorem 3.1, then it reduces to the following:

Corollary 3.3.
Let $\{A_i / i \in N\}$ be a finite set of non zero matrices in $P_n(\mathcal{F})$, then A = $\bigoplus_{i=1}^{k} A_i$ is idempotent if any one of the following conditions holds:

(i) $A_i A_j = A_j A_i = 0$

(ii) A_i's are pairwise comparable.

(iii) A_i's are pairwise space comparable.

Corollary 3.4.
$P_n(\mathcal{F})$ with either usual ordering (or) with space ordering A $\le$ B (or) A $\overset{s}{\le}$ B for A, B $\in$ $P_n(\mathcal{F})$ is a fuzzy vector space.

Proof. This follows from Theorem 3.1.

Remark 3.1.
We note that the conditions in Theorem 3.1 are only sufficient and not necessary. This is illustrated in the following.

Example 3.1.
Let A = $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$; B = $\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}$; C = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

A, B, C $\in$ $P_2(\mathcal{F})$. A + B + C = $\begin{bmatrix} 1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ $\in$ $P_2(\mathcal{F})$.

AB $\ne$ BA $\ne$ 0, AC $\ne$ CA $\ne$ 0, BC $\ne$ CB $\ne$ 0. Thus condition (i) fails. A and C are not comparable. Thus condition (ii) fails. Here AB = A; AC = A; BC = B BA $\ne$ A; CA $\ne$ A; CB = B. By Lemma 2.1, $\mathcal{R}$(A) $\subseteq$ $\mathcal{R}$(B); $\mathcal{R}$(A) $\subseteq$ $\mathcal{R}$(C); $\mathcal{R}$(B) $\subseteq$ $\mathcal{R}$(C) but $\mathcal{C}$(A) $\not\subseteq$ $\mathcal{C}$(B); $\mathcal{C}$(A) $\not\subseteq$ $\mathcal{C}$(C); $\mathcal{C}$ (B) $\subseteq$ $\mathcal{C}$ (C).

Thus condition (iii) fails.

Example 3.2.
In Theorem 3.1, the condition that each A_i is idempotent cannot be relaxed. This is illustrated in the following:

Let A = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\in$ $P_2(\mathcal{F})$; B = $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\notin$ $P_2(\mathcal{F})$.

Here A $\le$ B and A $\overset{s}{\le}$ B. However A + B = B $\notin$ $P_2(\mathcal{F})$ and Theorem fails.

Remark 3.2.
We provide an example to show that the conditions (ii) and (iii) are essential in Theorem 3.1. Consider

A = $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; B = $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$; C = $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

A $\le$ B, A $\le$ C but B and C are not comparable, $(B \not\le C; C \not\le B)$. A and C are not space comparable. Here, conditions (ii) and (iii) fail.

A + B + C = $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\notin$ $P_2(\mathcal{F})$. Thus theorem fails.

4. Product of Idempotent Matrices
Here, we shall derive conditions for the product of idempotent matrices to be idempotent. We show that $P_n(\mathcal{F})$ is a fuzzy algebra under certain conditions.

Theorem 4.1.
Let $\{A_i\}$ i $\in$ N be a finite set of idempotent matrices. Then A = $A_1 A_2 ... A_k$ is idempotent if any one of the following conditions holds for all $1 \le i < j \le k$.

(i) $A_i A_j = A_j A_i$ (ii) $\mathcal{R}(A_i) \subseteq \mathcal{R}(A_j)$
(iii) $\mathcal{R}(A_i) \supseteq \mathcal{R}(A_j)$ (iv) $\mathcal{C}(A_i) \subseteq \mathcal{C}(A_j)$
(v) $\mathcal{C}(A_i) \supseteq \mathcal{C}(A_j)$

Proof. Suppose (i) holds then A is idempotent.

Suppose (ii) holds then $\mathcal{R}(A_1) \subseteq \mathcal{R}(A_2) \subseteq \mathcal{R}(A_3) \subseteq ... \subseteq \mathcal{R}(A_k)$. Then by repeated applications of Lemma 2.2 it follows that A = A_1 hence A is idempotent.

Suppose (iii) holds then $\mathcal{R}(A_1) \supseteq \mathcal{R}(A_2) \supseteq ... \supseteq \mathcal{R}(A_k)$ then, again by lemma 2.2, A = A_k and hence A is idempotent.
(iv) and (v) can be proved in the same manner and hence omitted.

Remark 4.1.
It is clear that $P_n(\mathcal{F})$ is an idempotent fuzzy algebra under space ordering.

5. Applications.

In this section, by using the results on product of idempotent matrices of section 4, we get a factorization of a general idempotent Toeplitz matrix whose factors are idempotent matrices. Let us consider a Toeplitz matrix of order n of the form

$$T_n(a) = \begin{bmatrix} a & a_1 & a_2 & . & . & . & a_{n-1} \\ a_1 & a & a_1 & . & . & . & a_{n-2} \\ a_2 & a_1 & a & . & . & . & a_{n-3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n-3} & . & . & . & . & . & a_2 \\ a_{n-2} & a_{n-3} & a_{n-4} & . & . & a & a_1 \\ a_{n-1} & a_{n-2} & a_{n-3} & . & . & a_1 & a \end{bmatrix}$$

Lemma 5.1.

(i) Each $[a] \in [0, 1]$ is idempotent.

(ii) $T_2(a) \in P_2(\mathcal{F}) \Leftrightarrow a \geq a_1$.

(iii) $T_3(a) \in P_3(\mathcal{F}) \Leftrightarrow a \geq a_2 \geq a_1$.

Proof. (i) is trivial. (ii) is straightforward.

(iii) Let us partition $T_3(a) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where

$A = T_2(a)$; $C = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$; $B = C^T$; $D = [a]$

A is idempotent $\Leftrightarrow a \geq a_1$. By Theorem 5 of [3],

$L = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ is idempotent $\Leftrightarrow a \geq a_2 \geq a_1$.

By Theorem 4.1, $T_3(a) = LL^T = L^TL$ is idempotent $\Leftrightarrow$ L is idempotent $\Leftrightarrow a \geq a_2 \geq a_1$.

Theorem 5.2.

$T_n(a)$ is an idempotent matrix if

$a \geq a_{n-1} \geq a_{n-2} \geq ... \geq a_2 \geq a_1$.

Proof. $T_1(a)$, $T_2(a)$, $T_3(a)$ are all idempotent under the condition $a \geq a_2 \geq a_1$. Let us prove the Theorem by induction on n. Let us assume $T_{r-1}(a)$ is idempotent and prove that $T_r(a)$ is idempotent under the condition,

$\mathbf{a} \geq a_{r-1} \geq a_{r-2} \geq \ldots \geq a_2 \geq a_1$. Let us partition $T_r(a) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A = T_{r-1}(a)$,

$C = \begin{bmatrix} a_{r-1}, a_{r-2}, \ldots a_1 \end{bmatrix}$, $B = C^T$ and $D = [a]$.

By Theorem 6 of [3], $L = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ $L^T = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ are idempotent under the given condition. By Theorem 4.1, $T_r(a) = LL^T = L^TL$ is idempotent. ■

Theorem 5.3.

Let C_1 and C_2 be the concept matrices of any two Fuzzy Retrival systems with the same number of concepts. If C_1 and C_2 are comparable, then the transitive closures T_1 and T_2 are comparable. Further $T_1 T_2$ and $T_2 T_1$ are idempotent satisfying, any one of the following:

(i) $T_1 \leq T_1 T_2 \leq T_2$ and $T_1 \leq T_2 T_1 \leq T_2$

(ii) $T_2 \leq T_1 T_2 \leq T_1$ and $T_2 \leq T_2 T_1 \leq T_1$

Proof: By the definition of transitive closures [1],

$T_1 = C_1^p$ for some $p \leq n-1$ such that $C_1^{p+1} = C_1^p$,

$T_2 = C_2^q$ for some $q \leq n-1$ such that $C_2^{q+1} = C_2^q$,

Where n is the number of concepts of the systems.

Then it follows that when $p \leq q$, $C_1^q = C_1^p = T_1$

and when $q \leq p$, $C_2^p = C_2^q = T_2$.

Suppose $C_1 \leq C_2$, Then $C_1^p \leq C_2^p$ and $C_1^q \leq C_2^q$

Therefore $C_1 \leq C_2 \Rightarrow T_1 \leq T_2$. Similarly,
$C_2 \leq C_1 \Rightarrow T_2 \leq T_1$.

Thus the comparability of concept matrices is preserved for their transitive closures. The rest follows from the fact that both T_1 and T_2 are idempotent and from Theorem 4.1. ■

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