Formal averaging of quasi-periodic vector fields

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Work in progress with P. Chartier and J.M. Sanz-Serna

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Given $\omega \in \mathbb{R}^d$ non-resonant ($k \cdot \omega \neq 0$ for all $k \in \mathbb{Z}^d \backslash \{0\}$),

$$\frac{d}{dt} y = f(y, \omega t), \quad y(0) = y_0,$$

where $f(y, \theta)$ is

- smooth in y
- 2π-periodic in each component of $\theta \in \mathbb{R}^d$, with Fourier expansion

$$f(y, \theta) = \sum_{k \in \mathbb{Z}^d} e^{i(k \cdot \theta)} f_k(y).$$

- The map f itself may depend on the frequencies ω but we do not reflect that in the notation.

High order averaging of quasi-periodic vector fields (Perko 1969)

Given the quasi-periodic vector field

$$\frac{d}{dt}y = \epsilon f(y, \omega\, t) = \epsilon \sum_{k\in\mathbb{Z}^d} e^{i(k\cdot\omega)t} f_k(y).$$

there exists a formal quasi-periodic change of variables $y = K(Y, \omega\, t)$ that transforms the QP system into

$$\frac{d}{dt}Y = \epsilon F_1(Y) + \epsilon^2 F_2(Y) + \cdots$$

- The first term $F_1(y)$ is uniquely determined as

$$F_1(y) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(y,\theta)\, d\theta = f_0(y).$$

- $K(Y, \theta)$ is not unique. Classical choice:

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} K(y,\theta)\, d\theta = y$$

Efficient numerical integration of QP systems for large frequencies

- Symbolic-numeric algorithms using explicit knowledge of expansion of exact solution of QP systems
- Design and analyse methods to approximately integrate the smooth averaged system

$$\frac{d}{dt}Y = \epsilon F_1(Y) + \cdots + \epsilon^N F_N(Y), \quad Y(0) = y_0$$

instead of the highly oscillatory one by purely numerical schemes that try to approximate $Y(t)$ by using the original QPVF as input (Heterogeneous multiscale methods, ...).

Motivated by that, we aim at

- Obtaining formulae for solution of QP system, for averaged equations, solution of averaged equations, and change of variables.
- Such formulae should be as explicit as possible and of universal character

B-series expansion of solution of the QP system

For the solutions of $\dot{y} = \sum_k e^{i(k\cdot\omega)t} f_k(y)$,

$$y(t) = y(0) + \sum_{u\in\mathcal{T}} \frac{\alpha_u(t)}{\sigma_u} \mathcal{F}_u(y(0)),$$

$\mathcal{T}$ is the set of rooted trees labelled by $k \in \mathbb{Z}^d$, and for each $u \in \mathcal{T}$,

- the coefficients $\alpha_u(t)$ are linear combinations of $t^j e^{i(k\cdot\omega)t}$,
- the elementary differentials $\mathcal{F}_u : \mathbb{R}^d \to \mathbb{R}^d$ are smooth maps,

  ($\sigma_u \in \mathbb{Z}$ is a normalization factor.)

Elementary coefficients

$$\alpha_u(t) = \int_0^t e^{i(k\cdot\omega)t'} \alpha_{u_1}(t') \cdots \alpha_{u_m}(t')\, dt', \quad u = [u_1 \cdots u_m]_k.$$

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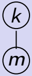
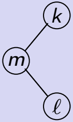
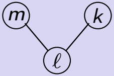
Examples for rooted trees with less than 4 vertices

u	$\mathcal{F}_u(y)$	$\alpha_u(t)$
	$f_k(y)$	$\int_0^t e^{i(k\cdot\omega)t_1}\,dt_1$
	$f'_m(y)f_k(y)$	$\int_0^t \int_0^{t_2} e^{i(kt_1+mt_2)\cdot\omega}\,dt_1\,dt_2$
	$f'_\ell(y)f'_m(y)f_k(y)$	$\int_0^t \int_0^{t_3} \int_0^{t_2} e^{i(kt_1+mt_2+\ell t_3)\cdot\omega}\,dt_1\,dt_2\,dt_3$
	$f''_\ell(y)(f_m(y),f_k(y))$	$\int_0^t \int_0^{t_2} e^{i(kt_1+mt_1+\ell t_2)\cdot\omega}\,dt_1\,dt_2$

For each $u \in \mathcal{T}$,

$$\alpha_u(t) = \sum_{k\in\mathbb{Z}^d} \alpha_u^k(t)e^{i(k\cdot\omega)t},$$

where each $\alpha_u^k(t)$ is a polynomial in t.

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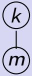
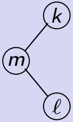
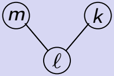
	$f'_m(y) f_k(y)$	$\int_0^t \int_0^{t_2} e^{i(kt_1+mt_2)\cdot\omega}\, dt_1\, dt_2$
	$f'_\ell(y) f'_m(y) f_k(y)$	$\int_0^t \int_0^{t_3} \int_0^{t_2} e^{i(kt_1+mt_2+\ell t_3)\cdot\omega}\, dt_1\, dt_2\, dt_3$
	$f''_\ell(y)(f_m(y), f_k(y))$	$\int_0^t \int_0^{t_2} e^{i(kt_1+mt_1+\ell t_2)\cdot\omega}\, dt_1\, dt_2$

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Averaging with B-series

There exist $\bar{\beta}_u, \bar{\alpha}_u(t), \kappa_u(\theta)$, $u \in \mathcal{T}$, ($\bar{\alpha}_u(t)$ polynomial, $\kappa_u(\theta)$ (2π)-periodic) such that for any solution $y(t)$ of the QP system

$$y(t) = K(Y(t), \omega t), \quad \frac{d}{dt} Y(t) = F(Y(t)),$$

where

$$\begin{aligned} F(Y) &= \sum_{u \in \mathcal{T}} \frac{\bar{\beta}_u}{\sigma_u} \mathcal{F}_u(Y), \\ Y(t) &= Y(0) + \sum_{u \in \mathcal{T}} \frac{\bar{\alpha}_u(t)}{\sigma_u} \mathcal{F}_u(Y(0)), \\ K(Y, \theta) &= Y + \sum_{u \in \mathcal{T}} \frac{\kappa_u(\theta)}{\sigma_u} \mathcal{F}_u(Y), \end{aligned}$$

That is, $\alpha(t) = \bar{\alpha}(t) * \kappa(t\omega)$ with $\frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t) * \bar{\beta}$.

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Observe that $\alpha(t)$ is such that

$$\frac{d}{dt}\alpha(t) = \alpha(t) * \beta(\omega t), \quad \alpha(0) = \mathbb{1},$$

where $\beta_k(\theta) = e^{ik\cdot\theta}$ and $\beta_u(\theta) = 0$ if u has more than one vertices. Recall that, for each $u \in \mathcal{T}$,

$$\alpha_u(t) = \sum_{k\in\mathbb{Z}^d} \alpha_u^k(t) e^{i(k\cdot\omega)t},$$

where each $\alpha_u^k(t)$ is a polynomial in t. Consider now

$$\gamma_u(t,\theta) = \sum_{k\in\mathbb{Z}^d} \alpha_u^k(t) e^{i(k\cdot\theta)},$$

so that $\alpha(t) = \gamma(t, t\omega)$. It is not difficult to see that

$$(\tfrac{\partial}{\partial t} + \omega \cdot \nabla_\theta)\gamma(t,\theta) = \gamma(t,\theta) * \beta(\theta), \quad \gamma(0,0) = \mathbb{1}.$$

Assume that there exist $\bar{\alpha}(t), \kappa(\theta) \in \mathcal{G}$, $\bar{\beta} \in \mathfrak{g}$ such that

$$\gamma(t,\theta) = \bar{\alpha}(t) * \kappa(\theta) \quad \text{with} \quad \frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t) * \bar{\beta}.$$

'Classical' averaging: $\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} K(Y,\theta)d\theta = Y \quad (Y(0) \neq y(0))$

We have for each $u \in \mathcal{T}$, $\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \kappa(\theta)_u d\theta = 0$ and thus,

$$\bar{\alpha}(t)_u = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \gamma_u(t,\theta)d\theta, \quad \kappa(\theta)_u = (\bar{\alpha}(0)^{-1} * \gamma(0,\theta))_u,$$

$$\bar{\beta}_u = \bar{\alpha}(0)^{-1} * \frac{d}{dt}\bar{\alpha}(t)_u\Big|_{t=0}.$$

Quasi-stroboscopic averaging: $K(Y,0) = Y \quad (Y(0) = y(0))$

Thus, $\kappa(0) = \mathbb{1}$ which implies $\bar{\alpha}(0) = \mathbb{1}$, so that

$$\bar{\alpha}(t) = \gamma(t,0), \quad \kappa(\theta) = \gamma(0,\theta), \quad \bar{\beta} = \frac{d}{dt}\bar{\alpha}(t)\Big|_{t=0}.$$

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But does the factorization

$$\gamma(t,\theta) = \bar{\alpha}(t) * \kappa(\theta), \quad \frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t) * \bar{\beta}, \tag{1}$$

of $\gamma(t,\theta)$ exist? Is it unique?

Theorem

If $\beta : \mathbb{T}^d \to \mathfrak{g}$ is such that, for each $u \in \mathcal{H}$, $\beta(\theta)_u$ is a trigonometric polynomial in θ, then

- *there exists a unique $\gamma : \mathbb{R} \times \mathbb{T}^d \to \mathcal{G}$ such that*

$$(\tfrac{\partial}{\partial t} + \omega \cdot \nabla_\theta)\gamma(t,\theta) = \gamma(t,\theta) * \beta(\theta), \quad \gamma(0,0) = \mathbb{1}.$$

- *there exist $\bar{\beta} \in \mathfrak{g}$ and $\kappa : \mathbb{T}^d \to \mathcal{G}$ such that (1) holds*
- $\kappa(\theta) = \kappa(0) * \gamma(0,\theta), \quad \bar{\alpha}(t) = \kappa(0)^{-1} * \gamma(t,0)$, *and*

$$\bar{\beta} = \kappa(0)^{-1} * \left(\frac{d}{dt}\gamma(t,0)\Big|_{t=0} \right) * \kappa(0).$$

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Quasi-stroboscopic averaging: $\kappa(0) = \mathbb{1}$.

In order to proof the theorem, we first observe that, if

$$\gamma(t,\theta) = \bar{\alpha}(t) * \kappa(\theta)$$

where

$$(\frac{\partial}{\partial t} + \omega \cdot \nabla_\theta)\gamma(t,\theta) = \gamma(t,\theta) * \beta(\theta), \quad \gamma(0,0) = \mathbb{1},$$

then

$$\begin{aligned} 0 &= \bar{\alpha}(t)^{-1} * (\frac{\partial}{\partial t} + \omega \cdot \nabla_\theta)(\gamma(t,\theta) - \bar{\alpha}(t) * \kappa(\theta)) \\ &= \kappa(\theta) * \beta(\theta) - \bar{\beta} \; * \kappa(\theta) - \omega \cdot \nabla_\theta \kappa(\theta). \end{aligned}$$

Sketch of proof: We first show that there exist unique $\kappa : \mathbb{T}^d \to \mathcal{G}$, $\bar{\beta} \in \mathfrak{g}$ such that

$$\omega \cdot \nabla_\theta \, \kappa(\theta) = \kappa(\theta) * \beta(\theta) - \bar{\beta} * \kappa(\theta), \quad \kappa(0) = \mathbb{1}.$$

Consider $\bar{\alpha}(t) = \exp^*(t \, \bar{\beta})$, so that

$$\frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t) * \bar{\beta}, \quad \bar{\alpha}(0) = \mathbb{1},$$

whence

$$(\tfrac{\partial}{\partial t} + \omega \cdot \nabla_\theta)\bar{\alpha}(t) * \kappa(\theta) = \bar{\alpha}(t) * \kappa(\theta) * \beta(\theta).$$

Finally, for $\gamma(t,\theta) := \bar{\alpha}(t) * \kappa(\theta)$,

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Expansion of solutions of quasi-periodic vector fields

$$y(t) = y_0 + \sum_{w \in \mathcal{W}} \alpha_w(t)\ f_w(y_0),$$

where $\mathcal{W}$ is the set of 'words' $w = k_1 \cdots k_r$ on the alphabet $\mathbb{Z}^d$,

$$\begin{aligned} \alpha_{k_1 \cdots k_r}(t) &= \int_0^t \int_0^{t_r} \cdots \int_0^{t_2} e^{i\omega\cdot(k_1 t_1 + \cdots + k_r t_r)}\, dt_1 \cdots dt_r, \\ f_{k_1 \cdots k_r}(y) &= \frac{\partial}{\partial y} f_{k_2 \cdots k_r}(y) f_{k_1}(y), \end{aligned}$$

In particular, $f_{mk} = f_k' f_m$, $f_{\ell m k} = f_k''(f_m, f_\ell) + f_k' f_m' f_\ell$.

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Indeed, consider for each $k \in \mathbb{Z}^d$, the Lie operator associated to f_k,

$$E_k = \sum_{i=1}^{d'} f_k^i \frac{\partial}{\partial y^i}, \quad \text{so that} \quad f_{k_1 \cdots k_r} = E_{k_1} \cdots E_{k_r}[\mathrm{id}],$$

Let Φ_t be such that $\Phi_t[g](y_0) = g(y(t))$ for smooth $g(y)$

$$\begin{aligned} \frac{d}{dt}\Phi_t[g](y_0) &= \frac{d}{dt} g(y(t)) = \frac{\partial}{\partial y} g(y(t)) f(y(t), \omega t) \\ &= \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\omega t} E_k[g](y(t)) = \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\omega t} \Phi_t E_k[g](y_0). \end{aligned}$$

Linear non-autonomous quasi-periodic differential equation

$$\frac{d}{dt}\Phi_t = \Phi_t \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\omega t} E_k, \quad \Phi_0 = I.$$

$$\Phi_t = I + \sum_{k_1 \cdots k_r \in \mathcal{W}} \alpha_{k_1 \cdots k_r}(t)\, E_{k_1} \cdots E_{k_r} \quad \text{(by Picard iteration)}$$

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$$\begin{aligned} \frac{d}{dt}\Phi_t[g](y_0) &= \frac{d}{dt} g(y(t)) = \frac{\partial}{\partial y} g(y(t)) f(y(t), \omega t) \\ &= \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\omega t} E_k[g](y(t)) = \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\omega t} \Phi_t E_k[g](y_0). \end{aligned}$$

Linear non-autonomous quasi-periodic differential equation

$$\frac{d}{dt}\Phi_t = \Phi_t \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\omega t} E_k, \quad \Phi_0 = I.$$

$$\Phi_t = I + \sum_{k_1 \cdots k_r \in \mathcal{W}} \alpha_{k_1 \cdots k_r}(t)\, E_{k_1} \cdots E_{k_r} \quad \text{(by Picard iteration)}$$

Recursive formulae for $\alpha_w(t)$

If $r \in \mathbb{Z}^+$, $k \in \mathbb{Z}^d - \{0\}$, $l \in \mathbb{Z}^d$, and $w \in \mathcal{W} \cup \{\emptyset\}$,

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\begin{aligned}
\alpha_k(t) &= \frac{i}{k \cdot \omega}(1 - e^{i(k\cdot\omega)t}), \\
\alpha_{0^r}(t) &= \frac{t^r}{r!}, \\
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\end{aligned}
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Actually, $\alpha_w(t) = \gamma_w(t, \omega t)$, where

$$\frac{\partial}{\partial t}\gamma_{wk}(t,\theta) + \omega \cdot \nabla_\theta \, \gamma_{wk}(t,\theta) = e^{ik\cdot\theta}\gamma_w(t,\theta), \quad \gamma_w(0,0) = 0.$$

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Formal high order averaging: Find a factorization

$$I + \sum_{w\in\mathcal{W}} \gamma_w(t,\theta)E_w \;=\; \Big(I + \sum_{w\in\mathcal{W}} \bar{\alpha}_w(t)\, E_w\Big)\Big(I + \sum_{w\in\mathcal{W}} \kappa_w(\theta)\, E_w\Big)$$

$$\frac{d}{dt}\Big(I + \sum_{w\in\mathcal{W}} \bar{\alpha}_w(t)\, E_w\Big) \;=\; \Big(I + \sum_{w\in\mathcal{W}} \bar{\alpha}_w(t)\, E_w\Big)\Big(\sum_{w\in\mathcal{W}} \bar{\beta}_w\, E_w\Big).$$

Let $\widehat{\mathcal{G}}$ (resp. $\widehat{\mathfrak{g}}$) the group of characters (resp. the Lie algebra of infinitesimal characers) of the shuffle Hopf algebra over $\mathcal{W}$. Given $\gamma : \mathbb{R} \times \mathbb{T}^d \to \widehat{\mathcal{G}}$, find $\bar{\beta} \in \widehat{\mathfrak{g}}$, $\bar{\alpha} : \mathbb{R} \to \widehat{\mathcal{G}}$, and $\kappa : \mathbb{T}^d \to \widehat{\mathcal{G}}$ such that

$$\gamma(t,\theta) = \bar{\alpha}(t) * \kappa(\theta), \quad \frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t) * \bar{\beta}.$$

Quasi-stroboscopic averaging: $\kappa(0) = \mathbb{1}$

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Recursion for coefficients of averaged equation

$$\begin{aligned}
\bar{\beta}_k &= 0, \quad \bar{\beta}_0 = 1, \quad \bar{\beta}_{0^{r+1}} = 0, \\
\bar{\beta}_{0^r k} &= \frac{i}{k \cdot \omega}(\bar{\beta}_{0^{r-1} k} - \bar{\beta}_{0^r} e^{i(k \cdot \theta_0)}), \\
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Quasi-stroboscopic averaging for quasi-periodic problems

We get $y(t) = K(Y(t), \omega t)$, where

$$\frac{d}{dt} Y = F(Y), \quad Y(0) = y_0,$$

with

$$\begin{aligned} K(Y,\theta) &= Y + \sum_{w \in \mathcal{W}} \kappa_w(\theta)\, f_w(Y), \\ F(Y) &= \sum_{w \in \mathcal{W}} \bar{\beta}_w\, f_w(Y), \\ Y(t) &= y_0 + \sum_{w \in \mathcal{W}} \bar{\alpha}_w(t)\, f_w(y_0). \end{aligned}$$

Since $\bar{\beta}$ is an infinitesimal character, $F(Y)$ is a Lie series, and thus shares the geometric properties of the vector fields f_k.

Concluding remarks

- We have introduced a new type of averaging of quasi-periodic vector fields that generalizes stroboscopic averaging of periodic vector fields. Such averaged VF is in the Lie algebra generated by the f_k in the Fourier expansion of the QPVF.
- We have shown that the averaged vector field by any other formal averaging is conjugate (by a time-independent near-to-identity map) to the quasi-stroboscopically averaged vector field.
- We have obtained neat expressions for the exact solution of QP systems, as well as for the averaged vector field, the solution of the averaged system, and the corresponding change of variables. Such expresions can be useful both computationally and for theoretical purposes.

Relation among coefficients of series indexed by words and trees

Consider a rooted tree $u \in \mathcal{T}$ of n vertices labelled by distinct $k_1, \ldots, k_n \in \mathbb{Z}^d$, and interpret u as a partial ordering of the set $\{k_1, \ldots, k_n\}$. Then,

$$\gamma_u(t, \theta) = \sum_{w \in \mathcal{W}_u} \gamma_w(t, \theta),$$

where $\mathcal{W}_u$ is the set of words of length n obtained by totally ordering $k_1, \ldots, k_n$ as an extension of the partial ordering of u.

Algebraic/geometric properties of exact solution

For $u_1, u_2, u_3 \in \mathcal{T}$

- $\gamma_{u_1 \circ u_2} + \gamma_{u_2 \circ u_1} = \gamma_{u_1} \gamma_{u_2}$, $\rightarrow$ symplectic for each fixed t, θ.
- In addition

$$\gamma_{u_1 \circ u_2 u_3} + \gamma_{u_2 \circ u_1 u_3} + \gamma_{u_3 \circ u_1 u_2} = \gamma_{u_1} \gamma_{u_2} \gamma_{u_3}.$$

  Hence, for each fixed t, θ, it represents the 1-flow of a vector field in the Lie algebra generated by the f_k.

- Classical averaging does not preserve the properties of $\gamma(t, \theta)$: Since $\bar{\alpha} = \langle \gamma \rangle$,

$$\bar{\alpha}_{u_1 \circ u_2} + \bar{\alpha}_{u_2 \circ u_1} \neq \bar{\alpha}_{u_1} \bar{\alpha}_{u_2}$$

  and likewise for the other equality in γ.
- Stroboscopic averaging: Properties inherited by $\bar{\alpha}(t) = \gamma(t, 0)$ (and $\bar{\beta}$). Averaged VF is in the Lie algebra of formal vector fields generated by f_k, $k \in \mathbb{Z}^d$.

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