S7. Copulas, Markov Operators and Mass Transportation

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- Carlo Sempi (Università del Salento, Italy)

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7. Wolfgang Trutschnig (University Salzburg, Austria)
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Multivariate copulas with hairpin support

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2010 Mathematics Subject Classification. 62H20; 60E05; 28A80

In this talk, we focus on the class of d–dimensional copulas ($d \geq 2$) whose support is contained in a hairpin set, which is, in dimension two, the union of the graphs of two increasing homeomorphisms h and g. Interestingly, however, the notion of two-dimensional hairpin allows for two different extensions to the general multivariate setting, which can be called sub-hairpin and super-hairpin, respectively.

We study existence and uniqueness of d–dimensional copulas whose support is contained in a sub- (or super-) hairpin and extend various results about doubly stochastic measures to the general multivariate setting.

In particular, we show that each copula with hairpin support is necessarily an extreme point of the convex set of all d–dimensional copulas and introduce a simple transformation that maps copulas with sub-hairpin support to copulas with super-hairpin support and vice–versa. Additionally, we calculate the corresponding Markov kernels and, using a simple analytic expression for sub- (or super-) hairpin copulas, analyze the strong interrelation with copulas having a fixed diagonal section. Several examples and graphics illustrate both the chosen approach and the main results.

Finally, we discuss the possible use of copulas with singular support (or with singular component) in applications, especially credit risk models.

The talk is based on a joint work with Juan Fernández-Sánchez and Wolfgang Trutschnig.

Copulas with fractal support

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2010 Mathematics Subject Classification. 60E05; 28A78; 28A80

Copulas with fractal support were introduced by Fredricks et al. in [3] where the authors demonstrated that for every value $s \in (1, d)$ it is possible to find a copula whose support has got this number as its Hausdorff dimension.

After that, there has been a increasing interest to study some properties of these copulas. In [1] de Amo et al. use some techniques of Ergodic Theory that allow to obtain new results. In particular, we studied the entropy of the associated dynamical system and some examples of copulas with the same fractal support whose associated measures are mutually singular. Also, it was started the study of self-similar copulas with full support that has been continued by Trutschnig and Fernández Sánchez in [5].

A natural question in Copula Theory is the generalization to higher dimensions of a bidimensional property. The case of the existence of $d-$copulas with fractal support of Hausdorff dimension equal to $s \in (1, d)$ was obtained by Trutschnig and Fernández Sánchez in [5].

Moreover, there are other fields of research that can be related with copulas as, for example, self-affine functions. This topic was analized by de Amo et al. in [2] where the authors characterize the self-affine functions whose graphs can be the support of a copula and find the Hausdorff dimension of the graph of these self-affine functions.

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Copulas of self-similar Ito diffusions

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This research was strongly influenced by the paper of C.Sempi (Sempi 2010) who was studying the possibility of coupling two Wiener processes by a given copula. We refine his approach. Namely, we study two-dimensional Ito diffusions (X, Y) whose margins are self-similar and are coupled by by a fixed copula C to a two-dimensional self-similar process. We characterize such copulas C in terms of differential equations. Furthermore we discuss their tail behaviour. Next we construct the Girsanov type change of measure which preserves the self-similarity. As an illustration we show that such change of measure may alter the copula of the process, i.e. we provide the example that the Roncalli proposition (see Coutant et al. 2001) cannot be extended for stochastic drifts.

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Univariate stochastic orders and joint stochastic orders: conditions on copulas for mutual relationships.

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In order to take into account any possible dependence between alternatives in optimization problems, bivariate characterizations of some well-know univariate stochastic orders have been defined and studied by Shanthikumar and Yao in 1991, and recently applied, e.g., in Belzunce et al. (2011), Li and You (2012) and Cai and Wei (2014). These characterizations gave rise to new stochastic comparisons, commonly called joint stochastic orders, which are equivalent to the original ones under assumption of independence, but are different whenever the variables to be compared are dependent. In this talk we describe sufficient conditions on the survival copula describing the dependence among the compared variables such that the standard stochastic orders imply the corresponding joint stochastic orders, and viceversa.

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General extremal dependence concepts

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The mathematical modeling of the relationship between *two or more* random variables calls for a notion of dependence. Dependence modeling is a fundamental task in several applied science and requires a challenging statistical analysis. While the concept of perfect positive dependence is not ambiguously defined for random vectors of arbitrary dimensions, various notions of perfect negative dependence arise when more than two random variables are involved. We survey all the concepts of negative dependence given in the literature and introduce novel generalised notions of perfect negative dependence which include them all as particular cases. Even if much of the literature on dependence is actually focused on positive dependence, we show that negative dependence plays an equally important role in the solution of many optimisation problems.

Design-risk and structural-risk in environmental applications: a multivariate approach

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The concept of Return Period is fundamental for the design and the assessment of many engineering and environmental works. In a multivariate framework, several approaches are available to its definition.

A consistent theoretical framework for the calculation of Return Periods in a multidimensional environment is outlined here, based on survival copulas and the corresponding survival Kendall's measures. This approach provides a coherent foundation of the notion of Return Period in a multivariate setting.

The notion of Return Period is commonly used in engineering and environmental applications to provide an estimate of the risk of failure of a structure. In this work we outline suitable procedures, based on the concept of Return Period, for computing both risky design events (concerning the random behavior of the geophysical environment where the structure of interest has to placed), and risky structural events (concerning the behavior of the structure of interest). As an illustration, practical hydrological applications are presented.

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Copulas from the Markov kernel perspective

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2010 Mathematics Subject Classification. 62H20; 60E05; 28A80

Using the one-to-one correspondence between two-dimensional copulas and Markov kernels having the Lebesgue measure on $[0,1]$ as fixed point allows for a translation of various well-known copula-related concepts to the Markov kernel setting and opens the door both to the definition of strong metrics and their induced dependence measures and to surprising mathematical aspects of copulas.

The talk will first sketch the construction of three metrics D_1, D_2, D_∞ on the space $\mathcal{C}$ of all two-dimensional copulas, mention their main properties and then show that these metrics, contrary to the standard uniform metric, strictly distinguish extreme kinds of statistical dependence. In order to illustrate the fact that translating copula-related concepts to the Markov kernel setting may simplify matters it will be shown that the shuffling operation translates to the well-known Frobenius-Perron operator and the star product to the standard composition of Markov kernels. Again using Markov kernels we will finally prove the existence of copulas exhibiting surprisingly singular behaviour - in fact, using tools from Symbolic Dynamics and Markov kernels once more, it is possible to construct singular copulas with full support for which all conditional distribution functions are continuous, strictly increasing, and have derivative zero almost everywhere.

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