S9. $\mathcal{D}$–modules and Singularities

Organizers:

- Andrea D'Agnolo (University of Padova, Italy)
- Francisco-Jesús Castro-Jiménez (University of Seville, Spain)
- Luis Narváez Macarro (University of Seville, Spain)

Speakers:

1. Takuro Abe (Kyoto University, Japan)
   Intersection points and Betti numbers of line arrangements
2. Tomoyuki Abe (University of Tokyo, Japan)
   Arithmetic $\mathcal{D}$-modules and Langlands correspondence
3. Alberto Castaño Domínguez (University of Seville, Spain)
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4. Andrea D'Agnolo (University of Padova, Italy)
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5. María Cruz Fernández Fernández (University of Seville, Spain)
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6. Michel Granger (University of Angers, France)
   Derivations of negative degree on quasihomogeneous isolated complete intersection singularities
7. Teresa Monteiro-Fernandes (University of Lisbon, Portugal)
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8. Giovanni Morando (Universities of Padova and Augsburg, Italy and Germany)
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   A functorial approach to asymptotic expansions
10. Kiyoshi Takeuchi (University of Tsukuba, Japan)
    Monodromies and asymptotic expansions at infinity of confluent A-hypergeometric functions

Intersection points and Betti numbers of line arrangements

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For a set of linear lines $\mathcal{A}_0$ in the real plane $\mathbf{R}^2$ with $|\mathcal{A}_0| =: n \geq 3$, consider the following problem. For a line $L \notin \mathcal{A}_0$, count $n_L := |\{\emptyset \neq H \cap L \mid H \in \mathcal{A}_0,\ H \neq L\}|$. It is clear that $n_L \in \{1, n-1, n\}$. This is equivalent to say that there are no lines $L \notin \mathcal{A}_0$ such that $1 < n_L < n-1$. Now recall that $t^2\mathrm{Poin}(M(\mathcal{A}_0), -t^{-1}) = t^2(1 + (-t^{-1}))(1 + (n-1)(-t^{-1})) = (t-1)(t-(n-1))$. Here $M(\mathcal{A}_0) = \mathbf{C}^2 \setminus \cup_{H \in \mathcal{A}_0} H$ is the complexified compliment of $\mathcal{A}_0$, and $\mathrm{Poin}(M(\mathcal{A}_0), t)$ its Poincaré polynomial.

From this viewpoint, we can generalize the above to all finite sets of affine lines in $\mathbf{R}^2$ (called affine line arrangements). Namely, let $\mathcal{A}$ be an affine line arrangement with $t^2\mathrm{Poin}(M(\mathcal{A}), -t^{-1}) = (t-\alpha)(t-\beta)$ such that α and β are real numbers with $\alpha \leq \beta$. Then there are no lines L such that $\alpha < n_L < \beta$. Moreover, if $n_L = \alpha$ or β, then the cone $c\mathcal{A}$ of $\mathcal{A}$ is a free arrangement.

The proof of this geometric (and actually combinatorial) inequality is based on algebraic geometry. Namely, we investigate the splitting type of the logarithmic vector bundles $D(c\mathcal{A})$ onto L. As a corollary, we prove that, for a line arrangement $\mathcal{A}$ and $H \in \mathcal{A}$, the freeness of the deletion pair $(c\mathcal{A}, c\mathcal{A} \setminus \{cH\})$ is determined by the combinatorics of $\mathcal{A}$.

Like this result, there seems to be some interesting relation between roots of Poincaré polynomials and geometry and combinatorics of line arrangements. If time permits, then we show the other relation between roots and the lower bound of chambers of real line arrangements.

[1] T. Abe, Exponents of 2-multiarrangements and multiplicity lattices. *J. Alg. Combin.* **35** (2013), 736–754.

[2] T. Abe, Roots of characteristic polynomials and intersection points of line arrangements (2013), http://arxiv.org/abs/1302.3822.

Arithmetic $\mathcal{D}$-modules and Langlands correspondence

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Over fields of characteristic $p > 0$, Grothendieck and his students established étale cohomology theory in 60's, which can be seen as an arithmetic analogue of singular cohomology theory, inspired by Weil conjecture. What is an analogue of de Rham cohomology theory in that context? Grothendieck also tackled this question, and defined the crystalline cohomology even though the theory is far from complete compare to étale cohomology. One of the problems was that the cohomology theory was defined only for proper and smooth varieties. To remedy this, Berthelot generalized to rigid cohomology theory, which is an analogue of de Rham cohomology for vector bundles with integrable connection. He further generalized this theory and introduced so called arithmetic $\mathcal{D}$-modules hoping that the rigid cohomology theory fits into the framework of the six functor formalism.

I will start from recalling the theory of Berthelot putting stress on the differences between arithmetic and complex $\mathcal{D}$-module theories, and explain how we can establish the six functor formalism for schemes of finite type over a field of characteristic $p > 0$ using results of Kedlaya, Caro and others. If time permits, I will talk about a proof of Langlands type correspondence for p-adic cohomology, or Deligne's "petits camarades cristallins" conjecture as an application.

Dwork families and $\mathcal{D}$-modules

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A Dwork family is a uniparametric monomial deformation of a Fermat hypersurface (cf. [1, § 2]). They were first studied by Bernard Dwork to understand the behaviour under a deformation of the zeta function of a hypersurface defined over a finite field. Recently, the interest on them has increased due to its connections to some problems from algebraic geometry, number theory and theoretical physics.

The invariant part of the Gauss-Manin cohomology of those families under the action of a group as in [1, § 3] has been studied in detail because of its connection with the L-function of Kloosterman sums. Following the development of Weil cohomologies, these works use techniques from ℓ-adic cohomology, p-adic analysis, or even complex differential geometry.

It would be desirable to have a p-adic algebraic description of that cohomology, in order to apply the result to the properties of Kloosterman sums which are invisible to other methods, such as finding the p-adic absolute values of the roots of the associated L-functions. As a first step in this direction, we present a way of carrying forward this work over $\mathbb{C}$ (or any algebraically closed field of characteristic zero) using algebraic $\mathcal{D}$-module theory.

This work is the main part of my doctoral dissertation, under the advising of Luis Narváez Macarro and Antonio Rojas León.

[1] Katz, N. M., Another look at the Dwork family, *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II* (Yuri Tschinkel and Yuri Zarhin ed.), Progress in Mathematics **270**, Birkhäuser, Boston, MA, 2009, 89–126.

Riemann-Hilbert correspondence for irregular holonomic $\mathcal{D}$-modules

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The classical Riemann-Hilbert correspondence establishes an equivalence between the triangulated category of regular holonomic $\mathcal{D}$-modules and that of constructible sheaves. In a joint work with Masaki Kashiwara, we prove a Riemann-Hilbert correspondence for holonomic $\mathcal{D}$-modules which are not necessarily regular. The construction of our target category is based on the theory of ind-sheaves by Kashiwara-Schapira and influenced by Tamarkin's work. Among the main ingredients of our proof is the description of the structure of flat meromorphic connections due to Mochizuki and Kedlaya. In this talk I will present an overview of the classical correspondence and the main ideas underlying our construction, sweeping under the carpet the more technical points.

L-characteristic cycles of A-hypergeometric D-modules

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We investigate the multiplicities of the irreducible components in the L-characteristic cycles of an A-hypergeometric D-module for any parameter vector, generalizing results from [3] and using techniques from [1]. We then apply this study to compute the dimension of the Gevrey solution spaces of A-hypergeometric D-modules for any parameter vector and compute new Gevrey solutions of A–hypergeometric D-modules, generalizing [2]. This is work in progress. Partially supported by MTM2010-19336 and FQM333.

[1] Berkesch, Ch., The rank of a hypergeometric system, *Compos. Math.* **147** (2011), 284–318.
[2] Fernández-Fernández, M., Irregular hypergeometric D-modules, *Adv. Math.* **224** (2010), 1735–1764.
[3] Schulze, M., Walther, U., Irregularity of hypergeometric systems via slopes along coordinate subspaces, *Duke Math. J.* **142** (2008), 465–509.

Derivations of negative degree on quasihomogeneous isolated complete intersection singularities

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J. Wahl conjectured that every quasihomogeneous isolated normal singularity admits a positive grading for which there are no derivations of negative weighted degree. In this talk we shall explain the motivations for this question and give a number of positive and negative results about it : we confirm the conjecture for quasihomogeneous isolated complete intersection singularities of either order at least 3 or embedding dimension at most 5. For each embedding dimension larger than 5 (and each dimension larger than 3), we give a counter-example to Wahl's conjecture.

Relative Riemann-Hilbert correspondence in dimension 1

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Let us consider a product $X \times S$ of complex manifolds and denote by $p : X \times S \to S$ the first projection. In a previous joint work with Claude Sabbah, motivated by the study of twistor $\mathcal{D}$-modules, we introduced the derived category of constructible $p^{-1}\mathcal{O}_S$-modules, and endowed it with a natural t-structure. So we obtained a natural notion of perversity over $p^{-1}\mathcal{O}_S$.

We proved that the derived functor Sol of holomorphic solutions takes the derived category of holonomic relative $\mathcal{D}_{X\times S/S}$-complexes, $D^b_{hol}(\mathcal{D}_{X\times S/S})$, to constructible objects, and takes strict (holonomic) modules to perverse objects with a perverse dual.

In this talk we construct the natural candidate to a quasi-inverse of Sol, and study it for $\dim X = \dim S = 1$, in which case it is completely understood.

This is part of a joint work in progress with Claude Sabbah.

$\mathcal{D}$–modules and sheaves on subanalytic sites

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In this talk we will expose some recent advances obtained on tempered solutions of holonomic D-modules with a particular emphasis on irregular D-modules and constructibility. The use of such complexes of sheaves on subanalytic sites is nowadays manifest. We will explain some techniques to work with these objects as well as some results obtained with their use, generalizing the classical ones in the regular case.

A functorial approach to asymptotic expansions

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Asymptotically developable expansions of holomorphic functions are an important tool to study differential equations with irregular singularities.

In this talk we will discuss the functorial nature of some kinds of asymptotics.

Monodromies and asymptotic expansions at infinity of confluent A-hypergeometric functions

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The theory of A-hypergeometric systems introduced by Gelfand-Kapranov-Zelevinsky is a vast generalization of that of classical hypergeometric differential equations. We call their holomorphic solutions A-hypergeometric functions. As in the classical case, A-hypergeometric functions admit Γ-series expansions and integral representations. Moreover they have deep connections with many other fields of mathematics, such as toric varieties, projective duality, period integrals, mirror symmetry, commutative algebra, enumerative algebraic geometry and combinatorics. Also from the viewpoint of the D-module theory, the corresponding (regular) holonomic D-modules were elegantly constructed by Gelfand-Kapranov-Zelevinsky. In 1994 Adolphson generalized A-hypergeometric systems and functions to the confluent (i.e. irregular) case. However in the confluent case, by the lack of the integral representation, almost nothing was known about the global properties of confluent A-hypergeometric functions. In this talk, we introduce their integral representation via Hien's rapid decay homology cycles and its applications. In particular, we give the formulas for their monodromies and asymptotic expansions at infinity.

This is a joint work with Alexander Esterov and Kana Ando.