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**NEUTRAL CANDIDATES IN APPROVAL
AND DISAPPROVAL VOTE**

by

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Neutral candidates in approval and disapproval vote*

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Abstract

In this article, the question is to select the “best” candidates within a set of candidates when voters cast approval-disapproval ternary ballots. That is, three options are offered to voters: casting a vote “in favor”, a “neutral” vote or a vote “against” each candidate. We first review desirable properties that a rule aggregating approval-disapproval ternary ballots should satisfy. We check whether the rules that have been proposed in the literature satisfy them. Then, we provide comparable axiomatizations of three rules: one is the lexicographical extension of the Approval rule for binary ballots; the second is the lexicographical extension of the Disapproval rule for binary ballots; and the third rule eliminates candidates with more opponents and fewer supporters than other candidates.

Keywords: Approval and Disapproval voting – Compromise – Condorcet principle.

1 Introduction

In elections very little information is generally asked to voters: those are generally asked to choose a unique candidate or unique party. Approval ballots (see Brams and Fishburn, 1978 or Weber, 1978) allow to vote in favor of several candidates. Boehm (1976) proposes in a mimeo (reported in Brams and Fishburn, 1978) what could be referred to disapproval vote, where voters could vote against candidates. Disapproval vote was used in the former Soviet Union for deputies elections: the rule introduced in 1987 gave voters the possibility to cross off the names of those against whom they wished to vote (see Hahn, 1988). The same rule is used in some Chinese village elections (see Zhong and Chen, 2002). For some other historical examples of rules that include negative options, see Kang (2010).

An approval ballot does not permit to distinguish between a neutral opinion and a negative one, and a disapproval ballot does not distinguish between a positive opinion and a neutral one. In an approval-disapproval ternary ballot, the three separate options are proposed: voters can cast a vote “in favor”, “neutral” or “against” each candidate. These are qualitative evaluations. Voters are not asked their cardinal preference on a numerical scale as it is the case with range voting (Smith, 2000) or evaluative voting (Hillinger, 2004, 2005, Smaoui and Lepelley, 2013). For a discussion of the difference between qualitative and quantitative evaluations see Balinski and Laraki (2014).

In this article, the question is to select the best candidate within a set of candidates when voters cast approval-disapproval ternary ballots. We first review desirable properties that a rule aggregating approval-disapproval ternary ballots should satisfy. Neutrality and Anonymity are standard properties. Concerning the latter property, we notice that behind the label of Anonymity what is often required is what we refer to as Anonymity for a given candidate. Anonymity simply requires that the name of the voters does not matter, while Anonymity for a given candidate requires that only the respective numbers of votes “in favor” or “against” a candidate matter.

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Next, we distinguish between rules where candidates are globally evaluated on the basis of their own merits and rules where candidates are compared. When a candidate is evaluated on the basis of his own merit, what is referred to as Independence in Rubinstein and Fishburn (1986), Kasher and Rubinstein (1997), Samet and Schmeidler (2003) may be required. This property states that the decision on each candidate should be made independently of other candidates. An example of a rule that satisfies this property is the Plurality rule defined by Ju (2005). In this rule a candidate is chosen if he receives more votes “in favor” than votes “against”.

When the rule deals with questions related to the acceptance of a candidate in a club, candidates can be evaluated on the basis on their merits. If the question is to select the best candidate within a set of candidates, those should not only be evaluated on the basis of their merits but also compared. That is, the relative position of one candidate compared to the others should also matter.

Voters with identical votes (be it, “in favor”, “neutral” or in “against”) on all candidates do not modify the relative positions of candidates. Therefore, these voters (defined in Brams and Fishburn, 1978, as unconcerned voters) should not change the result. Independence of unconcerned voters states that voters who cast an identical vote on all candidates can be disregarded. This property is close to the Contraction property in Aleskerov, Yakuba and Yuzbashev (2007) and implied by the Cancellation-independence in Gaertner and Xu (2012).

When candidates are compared, another principle to be used is the Pareto criterion. If, for each voter, a candidate is worse than another candidate, the worse candidate should not be selected, even if he or she exhibits good merits (for instance if he or she has more votes “in favor” than votes “against”). Here we require a stronger property that we refer to Independence of Pareto dominated candidates: if a candidate is worse than another candidate, not only this candidate should not be chosen, but the selection with his being present or not should not change the result. This property combines Independence of irrelevant candidates a la Nash (1950) and Pareto optimality. This property is similar to the Reduction property in Fishburn (1973).

The application of the Pareto criterion is sufficient to determine what happens to some unanimous candidates: candidates who only receive votes “in favor” should be selected and those who only receive votes “against” should be eliminated. The Pareto criterion partly applies to the unanimous candidates who only receives “neutral” votes, whom we refer to as neutral candidates. The Pareto criterion applies when neutral candidates face candidates with no vote “against”, or candidates with no vote “in favor”. But what should happen when a neutral candidate face candidates who have both votes “in favor” and votes “against”? Here there is a dilemma as neutral candidates have on the one hand, no vote “against” and, on the other hand, no vote “in favor”. We propose here three possible properties. The first property, named Choosing neutral candidates, solves the dilemma by selecting the neutral candidates. This property for a voting rule is close in spirit to the property of Non-compensatory threshold for social rankings introduced by Aleskerov, Yakuba and Yuzbashev (2007). The second property, Independence to dropping neutral candidates, solves the dilemma by choosing the “best” candidates among those who receive votes “against” and votes “in favor”. Choosing neutral candidates suggests to keep the neutral candidates, while Independence to dropping candidate suggests to select candidates with votes “in favor” and votes “against”. The third property, Compromise, suggests to solve the dilemma by selecting both the neutral candidates and the “best” candidates among those who receive votes “against” and votes “in favor”.

We check whether the rules that have been proposed in the literature (or that can be easily adapted to approval-disapproval ternary ballots) satisfy the properties. We focus on the Approval-Condorcet-Elimination procedure (Yilmaz, 1999), the Plurality rule (Ju, 2005) and the Dis&Approval rule (Alcantud and Laruelle, 2014), the Majority Judgement (Balinski and Laraki, 2007) and the Threshold social ranking (Aleskerov, Yakuba and Yuzbashev, 2007).

We provide comparable axiomatizations of three rules. The first one is the adaptation of the Threshold social ranking. It chooses the candidate with the smallest number of votes “against”. In case there are various candidates with an equal number of votes “against”, the candidates selected are the ones with the largest number of votes “in favor”. This rule is characterized by Anonymity for a given candidate, Neutrality, Independence of Pareto dominated candidates, Independence of unconcerned voters and Choosing neutral candidates. Substituting Choosing neutral candidates by Independence to dropping neutral candidates permits to characterize a second rule. This rule is the lexicographical generalization of the approval rule to ternary ballots. It chooses the candidate with the largest number of votes “in favor”, and in case of ties, the candidates with the smallest number of votes “against” are chosen. Substituting

Choosing neutral candidates by the Compromise property leads to the characterization of a third rule that we refer to as the Compromise rule. In this rule a candidate with fewer votes “in favor” and more votes “against” than another candidate is eliminated. The Compromise rule selects the candidates who are eliminated.

It will be shown that the set of candidates selected by the Compromise rule always contains the candidates selected by the other two rules characterized and by the Disapproval rule. The Compromise rule and the Plurality rule or the Median rule have at least one selected candidate in common. By contrast the Approval-Condorcet-Elimination procedure and the Compromise rule may select different candidates. In particular we provide an example where a candidate can at the same time obtain fewer supporters, more opponents than another candidate and be preferred by a majority of voters. This shows the clash between the Condorcet principle and the principle behind the Compromise rule.

The rest of the paper is organized as follows. Section 2 gives some definitions for approval-disapproval ternary ballots. Section 3 reviews the different properties to be requested for a rule selecting a candidate. Section 4 reviews four rules considered in the literature and checks their properties. Section 5 provides a characterization of three rules. Section 6 gives relations between the Compromise rule and the others. Section 7 concludes. Appendix contains all the proofs.

2 Ternary ballots

We consider voting situations where voters are asked to make a ternary choice on each available candidate, that can be “in favor”, “neutral” or “against”. A ternary ballot summarises a voter’s vote on all candidates, and a vote profile consists of the total collection of ballots. A voting rule associates to each vote profile a non empty subset of candidates.

Let \mathcal{N} be a countably infinite set, the universe of voters, and let \mathcal{C} be a countably infinite set, the universe of candidates. For each nonempty and finite set of n voters $N \subseteq \mathcal{N}$ and each nonempty and finite set of m candidates $C \subseteq \mathcal{C}$, a voting profile, denoted π_C^N , is a (n, m) matrix with entries in $\{-1, 0, 1\}$:

$$\pi_C^N = (a_{ip})_{\substack{i \in N \\ c_p \in C}}$$

where

$$a_{ip} = \begin{cases} 1, & \text{if voter } i \text{ casts a vote “in favor” of } c_p, \\ 0, & \text{if voter } i \text{ casts a “neutral” vote on } c_p, \\ -1, & \text{if voter } i \text{ casts a vote “against” } c_p. \end{cases}$$

Note that the values $\{-1, 0, 1\}$ are just an easy tool to represent the votes and comptabilize the respective numbers of votes of the three categories. No numerical values are associated to them. Let $\Pi_{n,m}(\{-1, 0, 1\})$ denote the set of (n, m) matrices, and Π be the set of all voting profiles that one can construct from \mathcal{N} and \mathcal{C} .

A candidate’s numbers of supporters and opponents are respectively the numbers of votes “in favor” and votes “against” received by the candidate. Candidate c_p ’s **number of supporters** is denoted $n_p^+(\pi_C^N)$, the **number of opponents** $n_p^-(\pi_C^N)$, while $n_p^0(\pi_C^N)$ denotes the number of voters who cast a “neutral” vote. That is,

$$\begin{aligned} n_p^+(\pi_C^N) &= \sum_{\substack{i \in N \\ a_{ip}=1}} a_{ip} \text{ and } n_p^-(\pi_C^N) = - \sum_{\substack{i \in N \\ a_{ip}=-1}} a_{ip}, \text{ while} \\ n_p^0(\pi_C^N) &= n - n_p^+(\pi_C^N) - n_p^-(\pi_C^N). \end{aligned}$$

A **positive candidate** has no opponent, a **negative candidate** has no supporter and a **neutral candidate** has neither opponent nor supporter. That is, c_p is a positive candidate if $n_p^-(\pi_C^N) = 0$; a negative candidate if $n_p^+(\pi_C^N) = 0$; a neutral candidate if $n_p^0(\pi_C^N) = n$. The remaining candidates are candidates with supporters and opponents: $n_p^+(\pi_C^N) \neq 0$ and $n_p^-(\pi_C^N) \neq 0$. For each $\pi_C^N \in \Pi$, denote by $\mathcal{C}_+(\pi_C^N)$ the set of positive candidates, $\mathcal{C}_-(\pi_C^N)$ the set of negative candidates, $\mathcal{C}_0(\pi_C^N)$ the set of neutral candidates, and by $\mathcal{C}_E(\pi_C^N)$ the set of candidates with supporters and opponents. Note that any of these sets can be empty, and $\mathcal{C}_0(\pi_C^N) = \mathcal{C}_+(\pi_C^N) \cap \mathcal{C}_-(\pi_C^N)$.

If $a_{ip} = a_{iq}$ voter i gives the same vote to candidates c_p and c_q (be it a vote “in favor”, a “neutral” vote or a vote “against”). If a voter gives the same vote to all candidates, then the voter is said to be unconcerned:¹ voter i is **unconcerned** if, for any pair $\{c_p, c_q\} \subseteq C$, it holds that $a_{ip} = a_{iq}$. If $a_{ip} \geq a_{iq}$, voter i considers that candidate c_p dominates candidate c_q . In a voting profile π_C^N , if all voters consider that candidate c_p **dominates** candidate c_q , then write $c_p \succsim_{\pi_C^N} c_q$. If $c_p \succsim_{\pi_C^N} c_q$ but $c_q \succsim_{\pi_C^N} c_p$ does not hold, then candidate c_p **Pareto dominates** candidate c_q and denote it $c_p \succ_{\pi_C^N} c_q$. If we compare the candidates on the basis of their supporters and opponents, another relation of domination can be defined. We will say that **candidate c_p dominates candidate c_q in terms of supporters and opponents** if c_p has more supporters and fewer opponents than c_q , and write it as $c_p \sqsupseteq_{\pi_C^N} c_q$. Formally $c_p \sqsupseteq_{\pi_C^N} c_q$ if $n_p^+(\pi_C^N) \geq n_q^+(\pi_C^N)$ and $n_p^-(\pi_C^N) \leq n_q^-(\pi_C^N)$. If $c_p \sqsupseteq_{\pi_C^N} c_q$ but $c_q \sqsupseteq_{\pi_C^N} c_p$ does not hold, then we say that c_p strictly dominates candidate c_q in terms of supporters and opponents, and write it as $c_p \sqsupset_{\pi_C^N} c_q$. Of course, if $c_q \succeq_{\pi_C^N} c_p$, then $c_q \sqsupseteq_{\pi_C^N} c_p$, while the reverse implication does not hold. Both relations $\succeq_{\pi_C^N}$ and $\sqsupseteq_{\pi_C^N}$ define preorders on the set of candidates $(C, \succeq_{\pi_C^N})$ and $(C, \sqsupseteq_{\pi_C^N})$.

A **voting rule** or simply a **rule** \mathcal{W} on Π associates to any possible voting profile $\pi_C^N \in \Pi$ a non empty subset of candidates $\mathcal{W}(\pi_C^N) \subseteq C$.

3 Properties for a voting rule

We are interested in voting rules satisfying Anonymity and Neutrality, which are standard properties in the literature. Recall that Anonymity requires that exchanging any two voters’ votes does not modify the selection of the candidates. Neutrality requires that exchanging the votes received by two candidates results in an exchange of these candidates in the selection. Exchanging candidates or voters are permutations within the set of voters or candidates. Let us denote by $\mathfrak{S}(B)$ the set of the permutations of the elements of B .

If we exchange two voters’ ballots this results in a permutation of the rows of the matrix. Formally, from $\pi_C^N \in \Pi$ and $\sigma \in \mathfrak{S}(N)$, define the matrix $\pi_C^{\sigma N}$ as follows:

$$\pi_C^{\sigma N} = (a_{\sigma(i)p})_{\substack{i \in N \\ c_p \in C}} \in \Pi.$$

A rule \mathcal{W} on Π satisfies **Anonymity** if the following holds:

$$\forall \pi_C^N \in \Pi, \forall \sigma \in \mathfrak{S}(N), \quad \mathcal{W}(\pi_C^{\sigma N}) = \mathcal{W}(\pi_C^N).$$

Most rules proposed in the literature satisfy a stronger form of Anonymity, that could be referred to as Anonymity for a given candidate. That is, a permutation of votes for a given candidate does not modify the result. From $\pi_C^N \in \Pi$, $c_q \in C$ and $\sigma_q \in \mathfrak{S}(N)$ define the matrix $\pi_C^{\sigma_q N}$ as follows:

$$\pi_C^{\sigma_q N} = (a_{ip}^{\sigma_q})_{\substack{i \in N \\ c_p \in C}} \in \Pi,$$

where

$$\forall i \in N, \quad a_{iq}^{\sigma_q} = a_{\sigma_q(i)q} \text{ and } \forall c_p \in C \setminus \{c_q\}, \forall i \in N, \quad a_{ip}^{\sigma_q} = a_{ip}.$$

A rule \mathcal{W} on Π satisfies **Anonymity for a given candidate** if the following holds:

$$\forall \pi_C^N \in \Pi, \forall c_q \in C, \forall \sigma_q \in \mathfrak{S}(N), \quad \mathcal{W}(\pi_C^N) = \mathcal{W}(\pi_C^{\sigma_q N}).$$

Remark 1. If Anonymity for a given candidate is required, then what matters is the number of supporters and opponents for each candidate. The selected candidates in any profile $\pi_C^N \in \Pi$ can be determined on the basis of $(n_p^+(\pi_C^N), n_p^-(\pi_C^N))_{c_p \in C}$.

Regarding Neutrality, from $\pi_C^N \in \Pi$ and $\sigma \in \mathfrak{S}(C)$, define the matrix $\pi_{\sigma C}^N \in \Pi$ as follows:

$$\pi_{\sigma C}^N = (a_{i\sigma(p)})_{\substack{i \in N \\ c_p \in C}} \in \Pi.$$

A rule \mathcal{W} on Π satisfies **Neutrality** if the following holds:

$$\forall \pi_C^N \in \Pi, \forall \sigma \in \mathfrak{S}(C), \quad \sigma(\mathcal{W}(\pi_C^N)) = \mathcal{W}(\pi_{\sigma C}^N).$$

¹Brams and Fishburn (1978) define unconcerned voters as those who are indifferent among all candidates. Here, they are those who cast an identical vote on all candidates.

Remark 2. Note that if $\pi_C^N \in \Pi$ is such that for each pair of distinct candidates $\{c_p, c_q\} \subseteq C$, it holds that, for any $i \in N$, $a_{ip} = a_{iq}$, then, by Neutrality, $c_p \in \mathcal{W}(\pi_C^N)$ if and only if $c_q \in \mathcal{W}(\pi_C^N)$.

In order to add desirable properties we have to be more precise about the type of voting rules that we are looking for. Indeed the type of rule may differ whether the question concerns the membership of a club or the selection of the “best” candidates.

When the question concerns the membership to a club, what matters is the candidate’s merits, not the comparison with others. A candidate can just be evaluated on the basis of her or his own merits and not compared with other candidates. This is what would require the Independence property first proposed in Rubinstein and Fishburn (1986) for binary profiles of votes. In our context this would mean that the only information needed to make a decision on a candidate would be the sets of voters who voted “in favor” and “against” this candidate, respectively. Ju (2005) requires a property that combines the Independence property and Anonymity. The property states that if a candidate has the same numbers of voters “in favor” and “against” in two different vote profiles, then the candidate cannot be selected in one profile and not in the other.

Here, we are interested in choosing the “best” candidate(s) among a given set of candidates. Therefore, candidates should not only be evaluated on the basis of their merits but also compared. A candidate may have the same numbers of votes “in favor” and vote “against” in two profiles and be selected because no other candidate is better in one case and not selected in the other case because others candidates are better.

When the question is to choose the best candidate(s) properties introducing the comparison of candidates have to be used. In comparisons what matters are the relative votes that the candidates received compared to the others. Unconcerned voters, who cast identical votes (be it, “in favor”, “neutral” or “against”) on all candidates, do not modify the relative positions of candidates. The following property, Independence of unconcerned voters, states that unconcerned voters can be disregarded.

A rule \mathcal{W} on Π satisfies **Independence of unconcerned voters** if, for each $\pi_C^N \in \Pi$ for which there exist $i \in N$ such that, for each pair $\{c_p, c_q\} \subseteq C$, $a_{ip} = a_{iq}$, it holds that $\mathcal{W}(\pi_C^N) = \mathcal{W}(\pi_C^{N \setminus \{i\}})$.

Another natural criterion is the Pareto principle: if a candidate is considered as worse than another candidate by all voters the worse candidate should not be selected.

A rule \mathcal{W} on Π satisfies **Pareto dominated candidates** if, for each $\pi_C^N \in \Pi$ such that there exist $c_q, c_p \in C$ where $c_q \succ_{\pi_C^N} c_p$, it holds that $c_p \notin \mathcal{W}(\pi_C^N)$.

Here, we require a stronger property: not only the worse candidate should not be chosen, but the selection with his being present or not should not change the result. We refer to this property, as the Independence of Pareto dominated candidates.

A rule \mathcal{W} on Π satisfies **Independence of Pareto dominated candidates** if, for each $\pi_C^N \in \Pi$ such that there exist $c_q, c_p \in C$ where $c_p \succ_{\pi_C^N} c_q$, it holds that $\mathcal{W}(\pi_C^N) = \mathcal{W}(\pi_{C \setminus \{c_q\}}^N)$.

This property is a combination of the Pareto dominated candidates and the Independence of irrelevant alternative à la Nash (1950). This latter property in our context would state that if a candidate is selected, then this candidate remains selected even some non selected (irrelevant) candidates withdraw. In the same spirit, Balinski and Laraki (2014) require what they refer to as the independence of irrelevant alternatives in ranking: if a candidate is ranked higher than another candidate, then the ranking between the candidates cannot be reversed whenever other candidates are added or dropped.

The previous properties permit to determine what happens to candidates on whom all voters cast a unanimous vote “in favor”: the rule selects a subset of these candidates; and what happens to candidates on whom all voters cast a unanimous vote “against”: these candidates should not be selected by the rule.² By contrast, what happens to neutral candidates, i.e. to elements of $\mathcal{C}_0(\pi_C^N)$, is not specified.

A rule that satisfies Independence of Pareto dominated candidates will select the neutral candidate if all other candidates are negative candidates. That is, if the set of candidates reduces to negative and

²Unless all voters are unconcerned and cast votes “against” over C . In this case, by default, all candidates would be selected.

neutral candidates: $C = \mathcal{C}_-(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$, the selected candidates are the neutral ones. By contrast, if there is at least a positive candidate, neutral candidates are not selected. If $\mathcal{C}_+(\pi_C^N) \neq \emptyset$ then no neutral candidate is selected. Nothing is specified whenever the other candidates are mixed candidates, i.e. $C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$.

Three principles can be used, the first one suggests to select the neutral candidates and the second suggest to drop them and choose among the remaining candidates. The last principle is a compromise between the first two, it recommends to select the neutral candidates and the “best” candidates among the remaining candidates.

A rule \mathcal{W} on Π satisfies **Choosing neutral candidates** if, for each π_C^N for which $C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$ where $\mathcal{C}_0(\pi_C^N) \neq \emptyset$ and $\mathcal{C}_E(\pi_C^N) \neq \emptyset$, it holds that $\mathcal{W}(\pi_C^N) = \mathcal{C}_0(\pi_C^N)$.

A rule \mathcal{W} on Π satisfies **Independence to dropping neutral candidates** if, for each π_C^N for which $C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$ where $\mathcal{C}_0(\pi_C^N) \neq \emptyset$ and $\mathcal{C}_E(\pi_C^N) \neq \emptyset$, it holds that $\mathcal{W}(\pi_C^N) = \mathcal{W}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$.

A rule \mathcal{W} on Π satisfies **Compromise** if, for each π_C^N for which $C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$ where $\mathcal{C}_0(\pi_C^N) \neq \emptyset$ and $\mathcal{C}_E(\pi_C^N) \neq \emptyset$, it holds that $\mathcal{W}(\pi_C^N) = \mathcal{C}_0(\pi_C^N) \cup \mathcal{W}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$.

The Compromise property states that as soon as there is a vote “against” a candidate (who also receives at least one vote “in favor”), this candidate cannot be considered as better than a neutral candidate.

4 Four existing rules

We review here four voting rules for ternary ballots that have been proposed in the literature and check which properties they satisfy. We focus on the Approval-Condorcet-Elimination rule (Yilmaz, 1999), the Plurality rule (Ju, 2005), the Disapproval rule (Alcantud and Laruelle, 2014) and the Median rule (Balinski, Laraki, 2007). Note that some of them have to be slightly adapted in order to be a rule as defined in Section 2 (in particular the rule has to select a non-empty set of candidates). At the end of the section Proposition 1 summarizes the properties satisfied by the four rules.

Yilmaz (1999) introduces the **Approval-Condorcet-Elimination** rule, that we denote \mathcal{W}^{ACE} . This rule selects the candidate(s) according to an iterative procedure. Given a profile $\pi_C^N \in \Pi$, for each pair of candidates, the number of votes for a candidate is increased by 1 whenever a voter considers that the candidate strictly dominates the other candidate. That is, for each pair of distinct candidates $\{c_p, c_q\}$ in profile π_C^N we compute the number of voters who consider that candidate c_p strictly dominates candidate c_q , that we denote $m_{pq}(\pi_C^N)$.

$$m_{pq}(\pi_C^N) = |\{i \in N, a_{ip} > a_{iq}\}|.$$

Similarly $m_{pq}(\pi_C^N)$ denotes the number of voters who consider that c_q strictly dominates candidate c_p . The iterative procedure is as follows:

1. If there exists c_p such that, for each $c_q \in C \setminus \{c_p\}$ we have $m_{pq}(\pi_C^N) > m_{qp}(\pi_C^N)$, then c_p is a Condorcet winner and so $\mathcal{W}^{ACE}(\pi_C^N) = \{c_p\}$;
2. else if there is a (unique) candidate $c_r \in C$ such that, for each $c_q \in C \setminus \{c_r\}$, $m_{qr}(\pi_C^N) > m_{rq}(\pi_C^N)$, then c_r is a Condorcet loser and is eliminated. Set $C = C \setminus \{c_r\}$, and go back to point 1;
3. else if $\arg \max_{q \in C} n_q^-(\pi_C^N) \neq C$, then the candidates belonging to this set are eliminated. Set $C = C \setminus \arg \min_{q \in C} n_q^-(\pi_C^N)$ and go back to point 1;
4. else $\mathcal{W}^{ACE}(\pi_C^N) = C$.

In words, the Approval-Condorcet-Elimination rule combines Condorcet criteria and the elimination of inferior alternatives. A candidate who receives a Condorcet majority over all others is declared a winner (point 1); if none exists, any Condorcet loser is eliminated (point 2); if there is no Condorcet candidate of either kind, candidates with the largest number of disapproval votes are eliminated and the Condorcet criteria are applied again except in the particular case where each candidate receives the same

number of disapproval votes (point 3). In the latter case, the entire set of candidates are considered as winners (point 4).

This rule satisfies Anonymity and Neutrality but does not satisfy the stronger Anonymity for a given candidate. Indeed as noted in Remark 1, this property requires that only the respective number of supporters and opponents of all candidates matter: $(n_p^+(\pi_C^N), n_p^-(\pi_C^N))_{c_p \in C}$. It is easy to construct examples of two different situations π_C^N and $\bar{\pi}_C^N$ with $n_p^+(\pi_C^N) = n_p^+(\bar{\pi}_C^N)$; $n_p^-(\pi_C^N) = n_p^-(\bar{\pi}_C^N)$ for any $c_p \in C$ but $\mathcal{W}^{ACE}(\pi_C^N) \neq \mathcal{W}^{ACE}(\bar{\pi}_C^N)$.

Example 1 Consider the following example with $C = \{c_1, c_2\}$ and $n = 3$:

$$\pi_C^N = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix} \text{ and } \bar{\pi}_C^N = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

In both profiles π_C^N and $\bar{\pi}_C^N$ we have, for each candidate $c_p \in C$, $n_p^+(\pi_C^N) = n_p^+(\bar{\pi}_C^N) = 1$; $n_p^-(\pi_C^N) = n_p^-(\bar{\pi}_C^N) = 1$. The difference between π_C^N and $\bar{\pi}_C^N$ is that the votes of voters 2 and 3 on candidate c_1 are permuted. In profile π_C^N , voter 2 is indifferent, voter 1 prefers candidate c_1 to c_2 while the reverse holds for voter 3: we have $m_{12}(\pi_C^N) = m_{21}(\pi_C^N) = 1$ and thus $\mathcal{W}^{ACE}(\pi_C^N) = \{c_1, c_2\}$. In profile $\bar{\pi}_C^N$, voter 1 prefers candidate c_1 to candidate c_2 , while voters 2 and 3 prefer candidate c_2 to candidate c_1 : $m_{21}(\bar{\pi}_C^N) = 2 > m_{12}(\bar{\pi}_C^N) = 1$ and thus $\mathcal{W}^{ACE}(\bar{\pi}_C^N) = \{c_2\}$.

This example illustrates that the Condorcet principle does not take into account the intensity of the vote: voter 1 largely prefers candidate c_1 (she votes “in favor” of c_1 and “against” c_2) while voters 2 and 3 slightly prefer candidate c_2 (“neutral” versus “against” and “in favor” versus “neutral” respectively).

Besides, we can expect that rule \mathcal{W}^{ACE} satisfies the Independence of unconcerned voters: \mathcal{W}^{ACE} is based on pairwise comparisons and adding or dropping voters who cast the same vote on all candidates does not affect the final result. This result is part of Proposition 1. We may have expected \mathcal{W}^{ACE} to satisfy the Pareto dominated candidates property. Surprisingly a Pareto dominated candidate can be selected as illustrated in Example 2.

Example 2 Consider the following example with $C = \{c_1, c_2, c_3, c_4\}$ and $n = 7$:

$$\pi_C^N = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix}$$

In this profile, there is no Condorcet winner, no Condorcet loser, and all candidates have the same numbers of opponents. In consequence, $\mathcal{W}^{ACE}(\pi_C^N) = \{c_1, c_2, c_3, c_4\}$. Candidate c_4 is selected although he is Pareto dominated by candidate c_3 .

Although a Pareto dominated candidate can be selected, we conjecture that the rule will never select a Pareto dominated candidate as single winner. The rule \mathcal{W}^{ACE} does not satisfy the Pareto dominated candidates property. Consequently, it cannot satisfy the Independence of dominated candidates. It does not satisfy either the properties concerning neutral candidates: the rule has not a systematic behavior towards these candidates.

Among the family of rules introduced by Ju (2005), there is the so-called **Plurality rule** \mathcal{W}^P in the context of ternary ballots. For each profile $\pi_C^N \in \Pi$, define the subset, possibly empty, of candidates

$$P(\pi_C^N) = \{c_p \in C : n_p^+(\pi_C^N) > n_p^-(\pi_C^N)\}.$$

The Plurality rule \mathcal{W}^P is defined as:

$$\mathcal{W}^P(\pi_C^N) = \begin{cases} P(\pi_C^N) & \text{if } P(\pi_C^N) \neq \emptyset, \\ C & \text{otherwise.} \end{cases}$$

A candidate is selected if his/her number of supporters is larger than his/her number of opponents. In case no candidate satisfies this condition all candidates are selected.

In rule \mathcal{W}^P the selection or not of a candidate is totally independent of the selection of the other candidates (unless no candidate satisfies the requirement). It is a rule that is to be used whenever the question is whether a candidate is suitable or not for a club membership, not for choosing the “best” candidate(s). Therefore, it will not be surprising that it does satisfy neither the Pareto dominated candidates property nor the Independence of unconcerned voters. There is no systematic treatment of the neutral candidates either.

What could be referred to as the **Median rule**, \mathcal{W}^{Med} , is the restriction of the majority judgement (Balinski and Laraki, 2007) to three labels, that allows for selecting more than one candidate. Given a profile $\pi_C^N \in \Pi$,

$$\mathcal{W}^{Med}(\pi_C^N) = \arg \max_{c_p \in C} m_p^n(\pi_C^N), \quad \text{where } m_p^n(\pi_C^N) = \begin{cases} 1 & \text{if } n_p^+(\pi_C^N) > n/2, \\ -1 & \text{if } n_p^-(\pi_C^N) > n/2, \\ 0 & \text{otherwise.} \end{cases}$$

Rule \mathcal{W}^{Med} is a rule that independently associates to each candidate her or his median evaluation, “in favor”, “neutral” or “against”. Then, it selects the subset of candidates whose median evaluation is “in favor”. If no candidate achieves this evaluation, it selects the subset of candidates whose median evaluation is neutral. Otherwise, all candidates are selected.

Adding or subtracting unconcerned voters may modify the median evaluation of a candidate (and not necessarily of others). Proposition 1 will confirm that this rule does not satisfy the Independence of unconcerned voters nor the Pareto dominated candidates criterion. The intuition for the last statement is that a candidate who is Pareto dominated may obtain the same median evaluation as the candidate who Pareto dominates her or him. If these two candidates appear to be in the highest evaluation class both of them will be selected. A neutral candidate will obtain the median evaluation “neutral”. This may be the highest category or not, depending on the profile.

Alcantud and Laruelle (2014) characterize the **Dis&approval rule** $\mathcal{W}^{D\&A}$. This rule assigns to each candidate a score which is the difference between the number of supporters and the number of opponents. It selects the candidate(s) with the largest score. Given a profile $\pi_C^N \in \Pi$

$$\mathcal{W}^{D\&A}(\pi_C^N) = \arg \max_{c_p \in C} (n_p^+(\pi_C^N) - n_p^-(\pi_C^N)).$$

Rule $\mathcal{W}^{D\&A}$ is a rule that independently associates to each candidate a global score which is the difference between the number of supporters and the number of opponents. Adding or subtracting unconcerned voters adds or subtracts an equal number to all candidates and does not modify the relative order of the scores. Therefore, we can expect that rule $\mathcal{W}^{D\&A}$ satisfies the Independence of unconcerned voters. This rule also satisfies the Independence of Pareto dominated candidates. Indeed, on the one hand, the score of a Pareto dominated candidate is smaller than the score of the one who Pareto dominates him or her and, on the other hand, the scores of the different candidates are independent of each other. A neutral candidate will obtain zero score. Compared with other candidates with both supporters and opponents, this score may be higher or lower, depending on the profile. No systematic property can be expected for neutral candidates in this rule.

The following proposition summarizes the results. The proof is given in the Appendix.

Proposition 1 *Let us consider the Approval-Condorcet-Elimination rule (\mathcal{W}^{ACE}), the Plurality rule (\mathcal{W}^P), the Median rule (\mathcal{W}^{Med}) and the Dis&Approval rule ($\mathcal{W}^{D\&A}$). We have the following results:*

- (i) **Anonymity and Neutrality** are satisfied by all the above mentioned rules. All these rules but \mathcal{W}^{ACE} also satisfy Anonymity for a given candidate;
- (ii) **Independence of unconcerned voters** is satisfied by \mathcal{W}^{ACE} and $\mathcal{W}^{D\&A}$. By contrast, \mathcal{W}^P and \mathcal{W}^{Med} do not satisfy this property;
- (iii) **Pareto dominated candidates and Independence of Pareto dominated candidates:** $\mathcal{W}^{D\&A}$ satisfies the Independence of Pareto dominated candidates (and thus Pareto dominated candidates) while \mathcal{W}^{ACE} , \mathcal{W}^P and \mathcal{W}^{Med} do not satisfy the Pareto dominated candidates;

- (iv) *Choosing neutral candidates, Independence to dropping neutral candidates and Compromises: no rule satisfies any of these three properties.*

5 Comparable axiomatic characterizations of three rules

One rule that has been proposed in the literature satisfies Choosing neutral candidates. It is the adaptation of the social ranking introduced by Aleskerov, Yakuba and Yuzbashev (2007) in the context of ternary ballots. This social ranking is defined as a binary relation over all candidates through a lexicographic criterion. Candidate c_p dominates candidate c_q either if c_p has strictly fewer opponents than c_q or, in case c_q and c_p have the same number of opponents, c_p has more supporters than c_q . We denote the corresponding voting rule by \mathcal{W}^{LD} and refer to it as the **Lexicographical disapproval** rule. Rule \mathcal{W}^{LD} selects the candidate who has the smallest number of opponents; ties among candidates are solved by choosing the candidate(s) with the largest number of supporters. Given a profile $\pi_C^N \in \Pi$,

$$\mathcal{W}^{LD}(\pi_C^N) = \arg \max_{c_p \in C^-(\pi_C^N)} (n_p^+(\pi_C^N)) \text{ where } C^-(\pi_C^N) = \arg \min_{c_p \in C} (n_p^-(\pi_C^N)).$$

Instead of minimizing the number of opponents, we could maximize the number of supporters and solving ties by choosing the candidate(s) with the smallest number of opponents. We denote this rule \mathcal{W}^{LA} and refer to it as **Lexicographical approval**. This is indeed a lexicographical extension of the binary disapproval rule to ternary ballots. Given a profile $\pi_C^N \in \Pi$,

$$\mathcal{W}^{LA}(\pi_C^N) = \arg \min_{c_p \in C^+(\pi_C^N)} (n_p^-(\pi_C^N)) \text{ where } C^+(\pi_C^N) = \arg \max_{c_p \in C} (n_p^+(\pi_C^N)).$$

The Lexicographical approval and the Lexicographical disapproval rules can also be seen as extensions of the Dis&approval rule. Let

$$\mathcal{W}^{s,t}(\pi_C^N) = \arg \max_{c_p \in C} (sn_p^+(\pi_C^N) - tn_p^-(\pi_C^N)) \text{ for some } s, t > 0.$$

The rule chooses the candidates whose weighted difference between his/her number of supporters and opponents is the largest. For each $\pi_C^N \in \Pi$, we have

$$\mathcal{W}^{LA}(\pi_C^N) = \mathcal{W}^{n+1,1}(\pi_C^N) \text{ and } \mathcal{W}^{LD}(\pi_C^N) = \mathcal{W}^{1,n+1}(\pi_C^N).$$

In rules $\mathcal{W}^{D\&A}$, \mathcal{W}^{LA} and \mathcal{W}^{LD} both numbers of supporters and opponents matter and there is a compensation between the number of supporters and the number of opponents. In rule $\mathcal{W}^{D\&A}$ both numbers are equally important, in rule \mathcal{W}^{LD} much more importance is given to the number of opponents; while in rule \mathcal{W}^{LA} much more importance is given to the number of supporters. So these rules differ in the importance given to both criteria, but there exists some compensation between the criteria. The question to be answered in order to choose among these rules is: what is more important, a vote from a supporter or a vote from an opponent? What we refer to as the Compromise rule, \mathcal{W}^{Co} , does not compare the two criteria. It only eliminates candidates who receive at the same time fewer votes “in favor” and more votes “against” than another candidate. The rule is based on the dominance relation in terms of supporters and opponents, $\sqsupseteq_{\pi_C^N}$. Given that $(C, \sqsupseteq_{\pi_C^N})$ is a finite preorder, the maximal elements forms a non empty subset of C . The **Compromise rule** \mathcal{W}^{Co} selects them. For each $\pi_C^N \in \Pi$,

$$\mathcal{W}^{Co}(\pi_C^N) = \{\text{maximal elements of } (C, \sqsupseteq_{\pi_C^N})\}.$$

Rules \mathcal{W}^{LD} , \mathcal{W}^{LA} and \mathcal{W}^{Co} satisfy Anonymity for a given candidate, Neutrality, Independence of unconcerned voters and the Independence of Pareto dominated candidates. A neutral candidate will obtain a score of zero in \mathcal{W}^{LD} and \mathcal{W}^{LA} . Given that what matters for \mathcal{W}^{LD} is the number of opponents, the neutral candidate is better than candidates with both opponents and supporters. The rule \mathcal{W}^{LD} satisfies Choosing neutral candidates. For \mathcal{W}^{LA} , what matters is the number of supporters; the neutral candidate is worse than candidates with both opponents and supporters. Moreover, dropping a neutral candidate does not affect the score of the other candidates: \mathcal{W}^{LA} satisfies Independence to dropping neutral candidates. It will be shown that \mathcal{W}^{Co} satisfies Compromise: a neutral candidate cannot be

eliminated because he or she has fewer opponents than candidates with both opponents and supporters; and candidates with both opponents and supporters are not dominated by the neutral candidate because they have more supporters. The main theorem of this paper tells us that these three rules can be characterized on the basis of these properties: the only property that distinguishes them is how neutral candidates are treated when they face candidates with both opponents and supporters.

Theorem 1 *Among the rules on Π satisfying Anonymity for a given candidate, Neutrality, Independence of Pareto dominated candidates and Independence to unconcerned voters,*

- (i) *the Compromise rule, \mathcal{W}^{Co} , is the only rule which satisfies Compromise;*
- (ii) *the Lexicographical disapproval rule, \mathcal{W}^{LD} , is the only rule which satisfies Choosing neutral candidates;*
- (iii) *the Lexicographical approval rule, \mathcal{W}^{LA} , is the only rule which satisfies Independence to dropping neutral candidates.*

Furthermore, the Compromise rule, \mathcal{W}^{Co} , is the largest rule with respect to set inclusion which satisfies Anonymity for a given candidate and Independence of Pareto dominated candidates.

The proof of Theorem 1 and the logical independence of the properties are demonstrated in the Appendix.

6 Choosing different winners?

The following question concerns the relation between the different rules in terms of selected candidates. It tells us that the Compromise rule is less selective than the Dis&Approval or any of its extensions, in particular the Lexicographical approval rule or the the Lexicographical disapproval rule.

Proposition 2 *For any $s, t > 0$, it holds that:*

$$\forall \pi_C^N \in \Pi, \quad \mathcal{W}^{Co}(\pi_C^N) \supseteq \mathcal{W}^{s,t}(\pi_C^N).$$

The proof is omitted for it is obvious. The drawback of the Compromise rule is that the number of eliminated candidates may be small. It is indeed easy to construct examples where no candidate is eliminated.

Example 3 Consider an instance $\pi_C^N \in \Pi$ with 5 candidates, $C = \{c_1, c_2, c_3, c_4, c_5\}$, and 250 voters such that:

	c_1	c_2	c_3	c_4	c_5
$n_p^+(\pi_C^N)$	90	130	115	150	145
$n_p^0(\pi_C^N)$	120	60	85	0	25
$n_p^-(\pi_C^N)$	40	60	50	100	80

We obtain $\mathcal{W}^{Co}(\pi_C^N) = \{c_1, c_2, c_3, c_4, c_5\}$. It can be said that each candidate has something in her or his favor: for instance c_1 is the candidate with the smallest number of opponents; c_4 is the candidate with the largest number of supporters. Each candidate can be selected by an extension of the Dis&Approval rule:

$$\begin{aligned} \mathcal{W}^{LD}(\pi_C^N) &= \{c_1\}, \\ \mathcal{W}^{D\&A}(\pi_C^N) &= \{c_2\}, \\ \mathcal{W}^{1,2}(\pi_C^N) &= \{c_3\}, \\ \mathcal{W}^{LA}(\pi_C^N) &= \{c_4\}, \\ \mathcal{W}^{2,1}(\pi_C^N) &= \{c_5\}. \end{aligned}$$

Therefore, it can be said that the Compromise property comes at a price: the number of selected candidates may increase. The number of selected candidates with the Compromise rule also reflects how difficult it is to choose among the candidates. It may be because the voters are very polarized in their votes or because candidates are very similar. Nevertheless, it could also be said that this rule avoids tyranny of a small majority. To see this, consider the following situation.

Example 4 The set N of voters contains one hundred elements and C contains three candidates. Pick any voting profile $\pi_C^N \in \Pi$ such that

$$\begin{array}{rcc} & c_1 & c_2 & c_3 \\ n_p^+(\pi_C^N) & 50 & 49 & 47 \\ n_p^0(\pi_C^N) & 0 & 3 & 6 \\ n_p^-(\pi_C^N) & 50 & 48 & 47 \end{array}$$

The three candidates are basically similar, and $\mathcal{W}^{Co}(\pi_C^N) = \{c_1, c_2, c_3\}$. Candidates differ for few votes, even if the Lexicographical rules and the Dis&Approval rule select different candidates:

$$\begin{aligned} \mathcal{W}^{LA}(\pi_C^N) &= \{c_1\}, \\ \mathcal{W}^{D\&A}(\pi_C^N) &= \{c_2\}, \\ \mathcal{W}^{LD}(\pi_C^N) &= \{c_3\}. \end{aligned}$$

Another result concerns the relation between the Compromise rule, on the one hand, and the Median rule or the Plurality rule, on the other hand. No inclusion relation holds as illustrated in Example 5.

Example 5 Consider the following instances where $C = \{c_1, c_2\}$ and $N = \{1, 2, 3\}$:

$$\pi_C^N = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \bar{\pi}_C^N = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$$

We have:

$$\begin{aligned} \mathcal{W}^{Co}(\pi_C^N) &= \{c_1\}, & \mathcal{W}^P(\pi_C^N) &= \mathcal{W}^{Med}(\pi_C^N) = \{c_1, c_2\}, \\ \mathcal{W}^{Co}(\bar{\pi}_C^N) &= \{c_1, c_2\} & \mathcal{W}^P(\bar{\pi}_C^N) &= \mathcal{W}^{Med}(\bar{\pi}_C^N) = \{c_2\}. \end{aligned}$$

Still the intersection of the selected candidates between the Compromise rule \mathcal{W}^{Co} and either the Plurality rule \mathcal{W}^P or the Median rule \mathcal{W}^{Med} is never empty as will be shown in Proposition 3 at the end of the section. By contrast the Compromise rule \mathcal{W}^{Co} and the Approval-Condorcet-Elimination rule \mathcal{W}^{ACE} can select different candidates as shown in Example 6.

Example 6 Let $\pi_C^N \in \Pi$ be a voting profile with two candidates, $C = \{c_1, c_2\}$, and seven voters:

$$\pi_C^N = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \\ -1 & 0 \\ -1 & 0 \end{pmatrix}$$

The number of supporters and opponents of the two candidates are respectively:

$$\begin{array}{rcc} & c_1 & c_2 \\ n_p^+(\pi_C^N) & 3 & 2 \\ n_p^0(\pi_C^N) & 2 & 2 \\ n_p^-(\pi_C^N) & 2 & 3 \end{array}$$

Thus $c_1 \sqsupset_{\pi_C^N} c_2$ with more supporters and fewer opponents, and so $\mathcal{W}^{Co}(\pi_C^N) = \{c_1\}$. The intersection with $\mathcal{W}^{ACE}(\pi_C^N)$ is empty. Indeed, $m_{12}(\pi_C^N) = 3$ and $m_{21}(\pi_C^N) = 4$, and so $\mathcal{W}^{ACE}(\pi_C^N) = \{c_2\}$.

This is an example where the Condorcet principle clashes with the domination relation: a candidate may be preferred by the majority of voters even if this candidate receives fewer votes “in favor” and more votes “against” than another candidate. The Condorcet principle takes into account the number of voters who prefer a candidate to the others, but does not take into account the intensity of the preference.

Proposition 3 summarizes the results of this section. Proof of (i) is given in the Appendix. Proof of (ii) is provided by Example 6 above.

Proposition 3 *Let us consider the Approval-Condorcet-Elimination rule (\mathcal{W}^{ACE}), the Plurality rule (\mathcal{W}^P), the Median rule (\mathcal{W}^{Med}) and the Compromise rule (\mathcal{W}^{Co}). The following holds:*

(i) *For any voting profile $\pi_C^N \in \Pi$ we have*

$$\mathcal{W}^{Co}(\pi_C^N) \cap \mathcal{W}^P(\pi_C^N) \neq \emptyset \quad \text{and} \quad \mathcal{W}^{Co}(\pi_C^N) \cap \mathcal{W}^{Med}(\pi_C^N) \neq \emptyset;$$

(ii) *There exists a voting profile $\pi_C^N \in \Pi$ such that*

$$\mathcal{W}^{Co}(\pi_C^N) \cap \mathcal{W}^{ACE}(\pi_C^N) = \emptyset.$$

7 Concluding remarks

The review of the properties makes clear a distinction between two different objectives: either accepting or not candidates on the basis of their exclusive merits or selecting the “best” candidates. In our opinion, the minimal properties for rules that select the “best” candidates are Neutrality, Anonymity, Independence of unconcerned voters and Pareto dominated candidates. Surprisingly, few rules proposed in the literature satisfy these properties. Among the discarded rules, only the Plurality rule satisfies the Independence property, which is a natural property for rules whose objective is to select candidates on the basis of their own merits.

Four rules satisfy these minimal properties for selecting the “best” candidates. These are the Disapproval rule, the Lexicographical disapproval, the Lexicographical approval and the Compromise rules. All these rules satisfy Anonymity for a given candidate. What matter to determine the selected candidates are the respective numbers of supporters and opponents. In three rules there is a compensation between the number of supporters and opponents. In the Compromise rule both criteria are taken into account without assigning a weight to any of them.

It may be worth studying rules that satisfy Anonymity but not Anonymity for a given candidate. Indeed, requiring Anonymity for a given candidate makes lose some information that may be relevant. Consider two voting profiles of 100 voters voting on 2 candidates. In both profiles, both candidates have 50 supporters and 50 opponents. Both candidates will be selected. But the profiles may be quite different. In one, 50 voters are supporters of the first candidate, and opponents of the second one, while the other 50 voters are opponents of the first candidate and supporters of the second. In this case, the selection of both candidates would reflect a polarized society. In the second profile, all voters are unconcerned, 50 voting “in favor” of both candidates and 50 voting “against” both candidates. In this case the selection of both candidates would reflect the fact that all voters are unconcerned. Similarly, voting rules that would satisfy Pareto dominated candidates but not the Independence of Pareto dominated candidates may be investigated.

Concerning the Compromise rule, some other remarks are worth noting. The compromise rule may be less selective than other rules. In theory, we can easily build example where the rule is little selective. There exists a trade-off between being selective and avoiding selecting non robust winner. In case the Compromise rule selects a single candidate, this means that the selected candidate is a very robust winner. The number of selected candidates with the Compromise rule also reflect how difficult it is to choose among the candidates. It may be because candidates are very close or because voters are very polarized. It would be worth testing in practice the Compromise rule.

This analysis has also shown the clash between the Condorcet principle and the evaluation of the candidates by voters. Basically, voters are not asked to rank candidates but to give their evaluation on them. So why should we expect the Condorcet principle to be satisfied?

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8 Appendix

8.1 Proof of Proposition 1

(i) **Anonymity and Neutrality.** The fact that \mathcal{W}^P , \mathcal{W}^{Med} , $\mathcal{W}^{D\&A}$ satisfy Neutrality and Anonymity for a given candidate must be clear from their respective definition. It must also be clear that \mathcal{W}^{ACE} satisfies Anonymity and Neutrality. Nevertheless, as shown in Exemple 1, \mathcal{W}^{ACE} violates Anonymity for a given candidate.

(ii) Independence of unconcerned voters. To show that $\mathcal{W}^{D\&A}$ satisfies Independence of unconcerned voters, consider a voting profile $\pi_C^N \in \Pi$ containing at least two voters and where there exists a voter $i \in N$ who casts, for each $c_q \in C$, $a_{iq} = 1$. Then, for each $c_q \in C$, $n_q^+(\pi_C^N) = n_q^+(\pi_C^{N \setminus \{i\}}) + 1$ and so

$$\arg \max_{c_p \in C} (n_p^+(\pi_C^N) - n_p^-(\pi_C^N)) = \arg \max_{c_p \in C} (n_p^+(\pi_C^{N \setminus \{i\}}) - n_p^-(\pi_C^{N \setminus \{i\}})).$$

From this, it follows that $\mathcal{W}^{D\&A}(\pi_C^N) = \mathcal{W}^{D\&A}(\pi_C^{N \setminus \{i\}})$. We proceed in a similar way in case there exists an unconcerned voter $i \in N$ who casts, for each $c_q \in C$, either $a_{iq} = -1$ or $a_{iq} = 0$.

To show that \mathcal{W}^{ACE} satisfies Independence of unconcerned voters, it suffices to observe that the mappings $m_{pq}, c_p, c_q \in C$, and $\arg \min_{r \in C} n_r^-$ are invariant under the operation of deletion of unconcerned voters.

To show that \mathcal{W}^P and \mathcal{W}^{Med} do not satisfy Independence of unconcerned voters, consider a voting profile $\pi_C^N \in \Pi$ containing an odd number $n > 1$ of candidates and at least three voters. Assume that there exist a voter $i \in N$ who casts, for each $c_q \in C$, $a_{iq} = 1$, a candidate $c_p \in C$ such that $n_p^+(\pi_C^N) = (n+1)/2$ and $n_p^-(\pi_C^N) = (n-1)/2$, and another candidate $c_r \in C$ such that $n_r^+(\pi_C^N) = n$. We obtain $\{c_p, c_r\} \subseteq \mathcal{W}^P(\pi_C^N)$, $c_p \notin \mathcal{W}^P(\pi_C^{N \setminus \{i\}}) \cup \mathcal{W}^{Med}(\pi_C^{N \setminus \{i\}})$ and $c_r \in \mathcal{W}^P(\pi_C^{N \setminus \{i\}}) \cap \mathcal{W}^{Med}(\pi_C^{N \setminus \{i\}})$, which is in contradiction with Independence of unconcerned voters.

(iii) Pareto dominated candidates and Independence of Pareto dominated candidates. To show that $\mathcal{W}^{D\&A}$ satisfies Independence of Pareto dominated candidates, consider a voting profile $\pi_C^N \in \Pi$ containing at least two distinct candidates $c_p \in C$ and $c_q \in C$ such that c_p Pareto dominates c_q , i.e. $c_p \succ_{\pi_C^N} c_q$. Therefore, for each voter $i \in N$, the following implications hold:

$$[a_{iq} = 1] \implies [a_{ip} = 1] \text{ and } [a_{ip} = -1] \implies [a_{iq} = -1].$$

This implies that $n_q^+(\pi_C^N) \geq n_p^+(\pi_C^N)$ and $n_q^-(\pi_C^N) \leq n_p^-(\pi_C^N)$. If $n_q^+(\pi_C^N) = n_p^+(\pi_C^N)$ and $n_q^-(\pi_C^N) = n_p^-(\pi_C^N)$, then $n_q^0(\pi_C^N) = n_p^0(\pi_C^N)$, which is in contradiction with the relation $c_p \succ_{\pi_C^N} c_q$. Thus, $n_q^+(\pi_C^N) \geq n_p^+(\pi_C^N)$ and $n_q^-(\pi_C^N) \leq n_p^-(\pi_C^N)$ with at least one strict inequality. By definition of the Dis&approval rule, we get $c_p \notin \mathcal{W}^{D\&A}(\pi_C^N)$ and, of course, $\mathcal{W}^{D\&A}(\pi_C^N) = \mathcal{W}^{D\&A}(\pi_C^N \setminus \{c_p\})$.

As shown in Example 2, \mathcal{W}^{ACE} does not satisfy Pareto dominated candidates.

To show that \mathcal{W}^P and \mathcal{W}^{Med} do not satisfy Pareto dominated candidates, consider a voting profile $\pi_C^N \in \Pi$ containing at least two candidates and three voters and where there exist $c_p, c_q \in C$ such that $n_p^+(\pi_C^N) = n$ and $n_q^+(\pi_C^N) = n-1$. Candidate c_q is Pareto dominated by candidate c_p but both candidates are contained in $\mathcal{W}^P(\pi_C^N)$ and $\mathcal{W}^{Med}(\pi_C^N)$, which is in contradiction with Pareto dominated candidates.

(iii) Choosing neutral candidates, Independence to dropping neutral candidates and Compromise. These three properties apply to voting profiles $\pi_C^N \in \Pi$ for which $C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$ where $\mathcal{C}_0(\pi_C^N) \neq \emptyset$ and $\mathcal{C}_E(\pi_C^N) \neq \emptyset$. Denote by $\Pi^* \subseteq \Pi$ such subset of voting profiles.

Pick any $n > 2$ and any $\pi_C^N \in \Pi^*$ where there is $c_p \in \mathcal{C}_E(\pi_C^N)$ such that $n_p^+(\pi_C^N) = n-1$. Then, $c_p \in \mathcal{W}^P(\pi_C^N)$ and $\mathcal{C}_0(\pi_C^N) \cap \mathcal{W}^P(\pi_C^N) = \emptyset$. Conclude that \mathcal{W}^P violates Choosing neutral candidates and Compromise. Next, pick any $n > 2$ and any $\pi_C^N \in \Pi^*$ where, for each $c_p \in \mathcal{C}_E(\pi_C^N)$, we have $n_p^-(\pi_C^N) = n-1$. In such a case, $\mathcal{W}^P(\pi_C^N) = C$, from which we conclude that \mathcal{W}^P violates Independence to dropping neutral candidates.

To show that $\mathcal{W}^{D\&A}$ does not satisfy Choosing neutral candidates, it suffices to look at the definition of $\mathcal{W}^{D\&A}$ and to remark that, in a voting profile $\pi_C^N \in \Pi^*$, the score $n_p^+(\pi_C^N) - n_p^-(\pi_C^N)$ of each candidate $c_p \in \mathcal{C}_0(\pi_C^N)$ is equal to zero, while the score $n_q^+(\pi_C^N) - n_q^-(\pi_C^N)$ of a candidate $c_q \in \mathcal{C}_E(\pi_C^N)$ can be positive, negative or equal to zero. In the first case, $\mathcal{C}_0(\pi_C^N) \cap \mathcal{W}^{D\&A}(\pi_C^N) = \emptyset$, and so Choosing neutral candidates is violated; in case the score of each $c_q \in \mathcal{C}_E(\pi_C^N)$ is negative, then $\mathcal{W}^{D\&A}(\pi_C^N) = \mathcal{C}_0(\pi_C^N)$ and so Independence of dropping neutral candidates and Compromise are violated.

To show that \mathcal{W}^{ACE} satisfy neither Choosing neutral candidates nor Independence to dropping neutral candidates nor Compromise, consider a voting profile $\pi_C^N \in \Pi^*$ where $\mathcal{C}_0(\pi_C^N) = \{c_1\}$, $\mathcal{C}_E(\pi_C^N) = \{c_2\}$. If $m_{12}(\pi_C^N) > m_{21}(\pi_C^N)$, then $\mathcal{W}^{ACE}(\pi_C^N) = \{c_1\}$ and so conclude that \mathcal{W}^{ACE} violates Compromise and Independence to dropping neutral candidates. If, instead, $m_{12}(\pi_C^N) < m_{21}(\pi_C^N)$, then $\mathcal{W}^{ACE}(\pi_C^N) = \{c_2\}$ and so conclude that Choosing neutral candidates is violated.

Proceeding in a similar way, we can easily show that \mathcal{W}^{Med} violates each of these three properties. This part is left to the reader. \square

8.2 Proof of Theorem 1

(Uniqueness) Pick any rule \mathcal{W} that satisfies Anonymity for a given candidate, Neutrality, Independence of unconcerned voters, Independence of Pareto dominated candidates on Π .

Pick any voting profile $\pi_C^N \in \Pi$. Denote by $\{c_1, c_2, \dots, c_m\}$ the list of candidates in C and by $\{1, \dots, n\}$ the set of voters in N . For $c_1 \in C$, there exists a permutation $\sigma_1 \in \mathfrak{S}(N)$ such that $i \geq j$ implies $a_{i1}^{\sigma_1} \geq a_{j1}^{\sigma_1}$. By Anonymity for a given candidate, $\mathcal{W}(\pi_C^N) = \mathcal{W}(\pi_C^{\sigma_1 N})$. Consider the resulting voting profile $\pi_C^{\sigma_1 N} \in \Pi$ and candidate $c_2 \in C$. There exists a permutation $\sigma_2 \in \mathfrak{S}(N)$ such that $i \geq j$ implies $a_{i2}^{\sigma_2 \sigma_1} \geq a_{j2}^{\sigma_2 \sigma_1}$. By Anonymity for a given candidate, $\mathcal{W}(\pi_C^{\sigma_1 N}) = \mathcal{W}(\pi_C^{\sigma_2 \sigma_1 N})$, and, by the previous step, $\mathcal{W}(\pi_C^N) = \mathcal{W}(\pi_C^{\sigma_2 \sigma_1 N})$. Proceeding in this way for each $c_q \in C$, we obtain a profile $\pi_C^{\sigma_m \dots \sigma_1 N} \in \Pi$ where the elements of each column are ranked in nondecreasing order. By Anonymity for a given candidate, we conclude that $\mathcal{W}(\pi_C^N) = \mathcal{W}(\pi_C^{\sigma_m \dots \sigma_1 N})$. For the sake of notation, from now on we will denote $\bar{\pi}_C^N$ the voting profile $\pi_C^{\sigma_m \dots \sigma_1 N}$ obtained from π_C^N . By construction, $\bar{\pi}_C^N$ is such that, for each candidate $c_p \in C$, the $n_p^+(\bar{\pi}_C^N)$ first voters cast a vote “in favor” of c_p , and the $n_p^-(\bar{\pi}_C^N)$ last voters cast a vote “against” c_p . From this observation, we can deduce that candidate $c_p \in C$ is Pareto dominated, with respect to $\succsim_{\bar{\pi}_C^N}$, by candidate $c_q \in C$ if and only if $n_q^+(\bar{\pi}_C^N) \geq n_p^+(\bar{\pi}_C^N)$ and $n_q^-(\bar{\pi}_C^N) \leq n_p^-(\bar{\pi}_C^N)$ with at least one strict inequality. Now, denote by $\bar{C} \neq \emptyset$ the subset of candidates in C that are not Pareto dominated in $\bar{\pi}_C^N$. By Independence of Pareto dominated candidates, $\mathcal{W}(\bar{\pi}_C^N) = \mathcal{W}(\bar{\pi}_{\bar{C}}^N)$. It remains to prove that $\mathcal{W}(\bar{\pi}_{\bar{C}}^N)$ is uniquely determined. If \bar{C} is a singleton, then $\mathcal{W}(\bar{\pi}_{\bar{C}}^N)$ is uniquely determined by the fact that a voting rule \mathcal{W} is nonempty-valued by definition. So, assume that \bar{C} contains at least two elements. Pick any two distinct candidates, c_p and c_q in \bar{C} . Two exclusive cases arise:

- (a) $n_p^+(\bar{\pi}_{\bar{C}}^N) = n_q^+(\bar{\pi}_{\bar{C}}^N)$ and $n_p^-(\bar{\pi}_{\bar{C}}^N) = n_q^-(\bar{\pi}_{\bar{C}}^N)$;
- (b) $n_p^+(\bar{\pi}_{\bar{C}}^N) > n_q^+(\bar{\pi}_{\bar{C}}^N)$ and $n_p^-(\bar{\pi}_{\bar{C}}^N) > n_q^-(\bar{\pi}_{\bar{C}}^N)$.

From (a) and (b) we can endow \bar{C} with a weak order (i.e. a complete and transitive relation) $R_{\bar{C}}$ defined as follows:

$$\forall c_p, c_q \in \bar{C}, \quad c_p R_{\bar{C}} c_q \quad \text{if} \quad [\text{either (a) or (b) holds}].$$

Therefore, the strict part of $R_{\bar{C}}$, denoted by $P_{\bar{C}}$, is as follows:

$$\forall c_p, c_q \in \bar{C}, \quad c_p P_{\bar{C}} c_q \quad \text{if} \quad [(b) \text{ holds}];$$

And the indifference part of $R_{\bar{C}}$, denoted by $I_{\bar{C}}$ is as follows:

$$\forall c_p, c_q \in \bar{C}, \quad c_p I_{\bar{C}} c_q \quad \text{if} \quad [(a) \text{ holds}].$$

Denote by $\bar{C}/I_{\bar{C}}$ the set of indifference classes of $(\bar{C}, R_{\bar{C}})$, and assume that the cardinality of $\bar{C}/I_{\bar{C}}$ equals $k \geq 1$. We can observe that the maximal elements of $(\bar{C}, R_{\bar{C}})$ belongs to the same indifference class denoted by [1]. Similarly, the maximal elements of $(\bar{C} \setminus [1], R_{\bar{C} \setminus [1]})$, where $R_{\bar{C} \setminus [1]}$ refers to the relation $R_{\bar{C}}$ restricted to $\bar{C} \setminus [1]$, are in the same indifference class denoted by [2]. Proceeding in this way until the class [k], we deduce that the indifference classes can be ordered as follows:

$$[1]R_{\bar{C}}^*[2]R_{\bar{C}}^* \dots R_{\bar{C}}^*[k-1]R_{\bar{C}}^*[k] \tag{1}$$

where $R_{\bar{C}}^*$ is the induced quotient relation (complete, transitive and antisymmetric) on $\bar{C}/I_{\bar{C}}$.

To complete the proof, we proceed by induction on the number of indifference classes. If $k = 1$, then, by Neutrality and the fact that \mathcal{W} is nonempty-valued, $\mathcal{W}(\bar{\pi}_{\bar{C}}^N) = [1]$ and so is uniquely determined, as desired. Next, assume that $\mathcal{W}(\bar{\pi}_{\bar{C}}^N)$ is uniquely determined for each $\bar{\pi}_{\bar{C}}^N \in \Pi$ such that $(\bar{C}, R_{\bar{C}})$ contains k indifference classes where $1 \leq k < r$. Pick any $\bar{\pi}_{\bar{C}}^N \in \Pi$ such that $(\bar{C}, R_{\bar{C}})$ contains $k = r$ indifference classes. By construction, for each $c_\ell \in [k]$, each $p \in \{1, \dots, k\}$ and each $c_q \in [p]$, we have

$n_\ell^+(\bar{\pi}_C^N) \leq n_q^+(\bar{\pi}_C^N)$ and $n_\ell^-(\bar{\pi}_C^N) \leq n_q^-(\bar{\pi}_C^N)$. Because the elements of each column of $\bar{\pi}_C^N$ are ranked in nondecreasing order, we can deduce that the first $n_\ell^+(\bar{\pi}_C^N)$ voters are unconcerned voters and vote “in favor” of each candidate, while the last $n_\ell^-(\bar{\pi}_C^N)$ voters are unconcerned voters who cast a vote “against” each candidate. Denote by $N' \subsetneq N$ this set of unconcerned voters who cast a vote “in favor” of each candidate or “against” each candidate in the voting profile $\bar{\pi}_C^N$. By Unconcerned voters, we deduce that $\mathcal{W}(\bar{\pi}_C^N) = \mathcal{W}(\bar{\pi}_C^{N \setminus N'})$. Because $\bar{C}/I_{\bar{C}}$ contains at least two indifference classes, one can observe that each candidate c_ℓ in $[k]$ receive the same positive number of “neutral” votes. Hence, we deduce that $\mathcal{C}_0(\bar{\pi}_C^{N \setminus N'})$ is nonempty and equal to $[k]$. Since $\mathcal{W}(\bar{\pi}_C^N) = \mathcal{W}(\bar{\pi}_C^{N \setminus N'})$, it remains to prove that $\mathcal{W}(\bar{\pi}_C^{N \setminus N'})$ is uniquely determined. To this end, we distinguish three exclusive cases.

(i) If \mathcal{W} satisfies Compromise, we have

$$\mathcal{W}(\bar{\pi}_C^{N \setminus N'}) = \mathcal{W}(\bar{\pi}_{\bar{C} \setminus [k]}^{N \setminus N'}) \cup \mathcal{C}_0(\bar{\pi}_C^{N \setminus N'}).$$

The weak order $(\bar{C} \setminus [k], R_{\bar{C} \setminus [k]})$, contains exactly $r - 1$ indifference classes. By the induction hypothesis, $\mathcal{W}(\bar{\pi}_{\bar{C} \setminus [k]}^{N \setminus N'})$ is uniquely determined, so is $\mathcal{W}(\bar{\pi}_C^{N \setminus N'})$.

(ii) If \mathcal{W} satisfies Choosing neutral candidates, then $\mathcal{W}(\bar{\pi}_C^{N \setminus N'}) = \mathcal{C}_0(\bar{\pi}_C^{N \setminus N'}) = [k]$, and so $\mathcal{W}(\bar{\pi}_C^{N \setminus N'})$ is uniquely determined.

(iii) If \mathcal{W} satisfies Independence to dropping neutral candidates, we have

$$\mathcal{W}(\bar{\pi}_C^{N \setminus N'}) = \mathcal{W}(\bar{\pi}_{\bar{C} \setminus [k]}^{N \setminus N'}).$$

The weak order $(\bar{C} \setminus [k], R_{\bar{C} \setminus [k]})$, contains exactly $r - 1$ indifference classes. By the induction hypothesis, $\mathcal{W}(\bar{\pi}_{\bar{C} \setminus [k]}^{N \setminus N'})$ is uniquely determined, so is $\mathcal{W}(\bar{\pi}_C^{N \setminus N'})$.

This completes the proof of the uniqueness part.

(*Existence*) We first show that \mathcal{W}^{Co} satisfies Anonymity for a given candidate, Neutrality, Independence of unconcerned voters, Independence of Pareto dominated candidates, and Compromise on Π .

Anonymity for a given candidate. For any voting profile $\pi_C^N \in \Pi$, any candidate $c_q \in C$ and any $\sigma_q \in \mathfrak{S}(N)$, it is immediate that $n_q^+(\pi_C^N) = n_q^+(\pi_C^{\sigma_q N})$ and $n_q^-(\pi_C^N) = n_q^-(\pi_C^{\sigma_q N})$, from which we deduce that $\mathcal{W}^{Co}(\pi_C^N) = \mathcal{W}^{Co}(\pi_C^{\sigma_q N})$.

Neutrality. For any voting profile π_C^N and any candidate $\sigma \in \mathfrak{S}(C)$, it is immediate that for each $c_q \in C$, $n_q^+(\pi_C^N) = n_{\sigma(q)}^+(\pi_{\sigma C}^N)$ and $n_q^-(\pi_C^N) = n_{\sigma(q)}^-(\pi_{\sigma C}^N)$, from which we deduce $\sigma(\mathcal{W}^{Co}(\pi_C^N)) = \mathcal{W}^{Co}(\pi_{\sigma C}^N)$.

Independence of unconcerned voters. Pick any voting profile $\pi_C^N \in \Pi$ where there is a voter $i \in N$ such that, for each pair of distinct candidates $c_p, c_q \in C$, we have $a_{ip} = a_{iq}$. Thus, we obtain:

$$\exists x \in \{0, 1\} \text{ such that } \forall c_p \in C, \quad n_p^+(\pi_C^N) = n_p^+(\pi_C^{N \setminus \{i\}}) + x \text{ and } n_p^-(\pi_C^N) = n_p^-(\pi_C^{N \setminus \{i\}}) + (1 - x).$$

We see that c_p is a maximal element of $(C, \supseteq_{\pi_C^N})$ if and only if c_p is a maximal element of $(C, \supseteq_{\pi_C^{N \setminus \{i\}}})$.

Therefore, by definition of \mathcal{W}^{Co} , we conclude that $\mathcal{W}^{Co}(\pi_C^N) = \mathcal{W}^{Co}(\pi_C^{N \setminus \{i\}})$.

Independence of Pareto dominated candidates. Pick any voting profile $\pi_C^N \in \Pi$ such that there exist two distinct candidates $c_q, c_p \in C$ where $c_p \succ_{\pi_C^N} c_q$. By definition of the relation $\succ_{\pi_C^N}$, for each $i \in N$, $a_{ip} \geq a_{iq}$ with a strict inequality for at least one $i \in N$. This implies that $n_p^+(\pi_C^N) \geq n_q^+(\pi_C^N)$ and $n_p^-(\pi_C^N) \leq n_q^-(\pi_C^N)$ with at least one strict inequality. From which, we conclude that $c_p \supseteq_{\pi_C^N} c_q$. Note that, for each $c_\ell \in C \setminus \{c_q\}$, $n_\ell^+(\pi_C^N) = n_\ell^+(\pi_{C \setminus \{c_q\}}^N)$ and $n_\ell^-(\pi_C^N) = n_\ell^-(\pi_{C \setminus \{c_q\}}^N)$. Therefore, c_ℓ is a maximal element of $(C, \supseteq_{\pi_C^N})$ if and only if it is a maximal element of $(C \setminus \{c_q\}, \supseteq_{\pi_{C \setminus \{c_q\}}^N})$,

and so $\mathcal{W}^{Co}(\pi_C^N) = \mathcal{W}^{Co}(\pi_{C \setminus \{c_q\}}^N)$.

Compromise. Pick any voting profile π_C^N such that $C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$ and where $\mathcal{C}_0(\pi_C^N) \neq \emptyset$ and $\mathcal{C}_E(\pi_C^N) \neq \emptyset$. To show: $\mathcal{W}^{Co}(\pi_C^N) = \mathcal{C}_0(\pi_C^N) \cup \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$. We proceed by double-inclusion.

1. First, we prove that $\mathcal{W}^{Co}(\pi_C^N) \subseteq \mathcal{C}_0(\pi_C^N) \cup \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$. Let c_p be a candidate belonging to $\mathcal{W}^{Co}(\pi_C^N)$.

If c_p belongs to $\mathcal{C}_E(\pi_C^N)$, then by definition of \mathcal{W}^{Co} , c_p necessarily belongs to $\mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$. Indeed, assume by way of contradiction that c_p belongs to $\mathcal{C}_E(\pi_C^N) \setminus \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$, then there exists $c_q \in \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$ such that $c_p \sqsupseteq_{\pi_C^N} c_q$. But since $\mathcal{C}_E(\pi_C^N) \subseteq C$, it is also true that $c_p \sqsupseteq_{\pi_C^N} c_q$ which contradicts the fact that $c_p \in \mathcal{W}^{Co}(\pi_C^N)$. Because $C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$, we deduce that $c_p \in \mathcal{C}_0(\pi_C^N) \cup \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$.

2. Second, we prove that $\mathcal{C}_0(\pi_C^N) \cup \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N) \subseteq \mathcal{W}^{Co}(\pi_C^N)$. Let $c_p \in \mathcal{C}_0(\pi_C^N) \cup \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$. Next, consider any candidate $c_q \in C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$. Four cases arise:

2.1 If $c_q \in \mathcal{C}_0(\pi_C^N)$ and $c_p \in \mathcal{C}_0(\pi_C^N)$, then $n_q^+(\pi_C^N) = n_p^+(\pi_C^N) = 0$ and $n_q^-(\pi_C^N) = n_p^-(\pi_C^N) = 0$ and it is immediate that $c_p \sqsupseteq_{\pi_C^N} c_q$.

2.2 If $c_q \in \mathcal{C}_0(\pi_C^N)$ and $c_p \in \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$ then $n_q^+(\pi_C^N) = 0 < n_p^+(\pi_C^N)$ which implies $c_q \not\sqsupseteq_{\pi_C^N} c_p$.

2.3 If $c_q \in \mathcal{C}_E(\pi_C^N)$ and $c_p \in \mathcal{C}_0(\pi_C^N)$, then $n_p^-(\pi_C^N) = 0 < n_q^-(\pi_C^N)$ which implies $c_q \not\sqsupseteq_{\pi_C^N} c_p$.

2.4 If $c_q \in \mathcal{C}_E(\pi_C^N)$ and $c_p \in \mathcal{W}^{Co}(\pi_{\mathcal{C}_E(\pi_C^N)}^N)$, then by definition of \mathcal{W}^{Co} , it holds that $c_p \sqsupseteq_{\pi_C^N} c_q$.

From the above four cases, we conclude that c_p belongs to $\mathcal{W}^{Co}(\pi_C^N)$.

We now show that \mathcal{W}^{LD} satisfies Anonymity for a given candidate, Neutrality, Independence of unconcerned voters, Independence of Pareto dominated candidates, and Compromise on II.

Anonymity for a given candidate and Neutrality. The proof is similar to the one given above for \mathcal{W}^{Co} and so is omitted.

Independence of unconcerned voters. Pick any voting profile $\pi_C^N \in \Pi$ containing at least two voters and where there exists a voter $i \in N$ who casts, for each $c_q \in C$, $a_{iq} = -1$. Then, we have

$$C^-(\pi_C^N) \stackrel{\text{def}}{=} \arg \max_{c_p \in C_p} (n_p^-(\pi_C^N)) = \arg \max_{c_p \in C_p} (n_p^-(\pi_C^{N \setminus \{i\}})) \stackrel{\text{def}}{=} C^-(\pi_C^{N \setminus \{i\}}),$$

and so $\mathcal{W}^{LD}(\pi_C^N) = \mathcal{W}^{LD}(\pi_C^{N \setminus \{i\}})$, as desired.

Next, suppose there exists a voter $i \in N$ who casts, either $a_{iq} = 1$ for each $c_q \in C$, or $a_{iq} = 0$ for each $c_q \in C$. Obviously, $C^-(\pi_C^N) = C^-(\pi_C^{N \setminus \{i\}})$, and, for each $c_q \in C^-(\pi_C^{N \setminus \{i\}})$, either $n_q^+(\pi_C^N) = n_q^+(\pi_C^{N \setminus \{i\}}) + 1$ or $n_q^+(\pi_C^N) = n_q^+(\pi_C^{N \setminus \{i\}})$. It follows that

$$\arg \max_{c_r \in C^-(\pi_C^N)} (n_r^+(\pi_C^N)) = \arg \max_{c_r \in C^-(\pi_C^{N \setminus \{i\}})} (n_r^+(\pi_C^{N \setminus \{i\}})),$$

and so $\mathcal{W}^{LD}(\pi_C^N) = \mathcal{W}^{LD}(\pi_C^{N \setminus \{i\}})$, as desired.

Independence of Pareto dominated candidates. Consider a voting profile $\pi_C^N \in \Pi$ where candidate $c_p \in C$ is Pareto dominated by another candidate $c_q \neq c_p$. Two cases arise. If c_p does not belong to

$$C^-(\pi_C^N) \stackrel{\text{def}}{=} \arg \min_{c_r \in C} (n_r^-(\pi_C^N)),$$

then $c_p \notin \mathcal{W}^{LD}(\pi_C^N)$ and $\mathcal{W}^{LD}(\pi_C^N) = \mathcal{W}^{LD}(\pi_{C \setminus \{c_p\}}^N)$ where the equality follows from the definition of the Lexicographical disapproval rule. If c_p belongs to $C^-(\pi_C^N)$, observe that $c_q \in C^-(\pi_C^N)$. Because

$n_q^-(\pi_C^N) = n_p^-(\pi_C^N)$ and c_q Pareto dominates c_p , we necessarily have $n_q^+(\pi_C^N) > n_p^+(\pi_C^N)$. Therefore, $c_p \notin \mathcal{W}^{LD}(\pi_C^N)$ and $\mathcal{W}^{LD}(\pi_C^N) = \mathcal{W}^{LD}(\pi_{C \setminus \{c_p\}}^N)$.

Choosing neutral candidates. Consider a voting profile $\pi_C^N \in \Pi$ such that $C = \mathcal{C}_E(\pi_C^N) \cup \mathcal{C}_0(\pi_C^N)$ and where $\mathcal{C}_0(\pi_C^N) \neq \emptyset$ and $\mathcal{C}_E(\pi_C^N) \neq \emptyset$. Because the number of votes “against” received by a candidate in $\mathcal{C}_E(\pi_C^N)$ is always positive, we can conclude that $\arg \min_{c_p \in C} n^-(\pi_C^N) = \mathcal{C}_0(\pi_C^N)$. By definition of the Lexicographical disapproval rule, we easily conclude that $\mathcal{W}^{LD}(\pi_C^N) = \mathcal{C}_0(\pi_C^N)$, as desired.

To prove that \mathcal{W}^{LA} satisfies Anonymity for a given candidate, Neutrality, Independence of unconcerned voters, Independence of Pareto dominated candidates, and Independence to dropping candidates on Π , we proceed in a similar way as above for \mathcal{W}^{LD} . The details are left to the reader. This concludes the existence part.

It remains to show that \mathcal{W}^{Co} is the largest rule with respect to set inclusion which satisfies Anonymity for a given candidate and Independence of Pareto dominated candidates. By the existence part, \mathcal{W}^{Co} satisfies these two properties. By the uniqueness part, if a rule satisfies Anonymity for a given candidate and Independence of Pareto dominated candidates, then, for each $\pi_C^N \in \Pi$, this rule selects a subset of $\bar{C} \neq \emptyset$, i.e. the subset of candidates in C that are not Pareto dominated in $\bar{\pi}_C^N$ (see the uniqueness part). By combining point (i) of the uniqueness part with the existence part, we have:

$$\mathcal{W}^{Co}(\bar{\pi}_C^N) = \mathcal{W}^{Co}(\bar{\pi}_{\bar{C}}^{N \setminus N'}) \quad \text{and} \quad \mathcal{W}^{Co}(\bar{\pi}_{\bar{C}}^{N \setminus N'}) = \mathcal{W}^{Co}(\bar{\pi}_{\bar{C} \setminus [k]}^{N \setminus N'}) \cup \mathcal{C}_0(\bar{\pi}_{\bar{C}}^{N \setminus N'}).$$

Using an inductive argument from $\bar{\pi}_{\bar{C} \setminus [k]}^{N \setminus N'}$, we conclude that $\mathcal{W}^{Co}(\bar{\pi}_{\bar{C}}^{N \setminus N'}) = \bar{C}$. \square

8.3 Logical independence of the properties

The logical independence of the properties used to characterize \mathcal{W}^{Co} can be demonstrated as follows:

1. For each $\pi_C^N \in \Pi$, let $\mathcal{D}(\pi_C^N)$ be the set of candidates $c_p \in C$ such that $n_p^+(\pi_C^N) = 1$. Let \mathcal{W} be the rule defined as follows: $\mathcal{W}(\pi_C^N) = \mathcal{W}^{Co}(\pi_C^N) \setminus \mathcal{D}(\pi_C^N)$ if $\mathcal{W}^{Co}(\pi_C^N) \not\subseteq \mathcal{D}(\pi_C^N)$, and $\mathcal{W}(\pi_C^N) = \mathcal{W}^{Co}(\pi_C^N)$ otherwise.

The rule \mathcal{W} is clearly different from \mathcal{W}^{Co} . For instance, consider the following voting profile where $C = \{c_1, c_2\}$. We have

$$\mathcal{W} \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} \right] = \{c_2\} \quad \text{and} \quad \mathcal{W}^{Co} \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} \right] = \{c_1, c_2\}.$$

The rule \mathcal{W} satisfies Anonymity for a given candidate, Neutrality, Independence of Pareto dominated candidates and Compromise, but violates Independence of unconcerned voters. For instance, consider the above example. The first voter is indifferent between c_1 and c_2 but

$$\mathcal{W} \left[\begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} \right] = \{c_1\} \neq \mathcal{W} \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} \right] = \{c_2\}.$$

2. The constant rule \mathcal{W} defined, for each $\pi_C^N \in \Pi$, as $\mathcal{W}(\pi_C^N) = C$, satisfies Anonymity for a given candidate, Neutrality, Independence of unconcerned voters and Compromise, but violates Independence of Pareto dominated candidates.
3. By point (ii) of Theorem 1, the Lexicographical disapproval rule \mathcal{W}^{LD} violates Compromise but satisfies Anonymity for a given candidate, Neutrality, Independence of unconcerned voters and Independence of Pareto dominated candidates.

4. Consider the rule \mathcal{W} which associates to each $\pi_C^N \in \Pi$, the set of maximal elements of the relation $(C, \succeq_{\pi_C^N})$. It must be clear that \mathcal{W} satisfies Neutrality Independence of unconcerned voters, Independence of Pareto dominated candidates and Compromise, but violates Anonymity for a given candidate.
5. Pick some c^* in C and let \mathcal{W} be the rule defined as follows: for each $\pi_C^N \in \Pi$ where $c^* \in C$, if no one casts a “neutral” vote for c^* and $\mathcal{W}^{Co}(\pi_C^N) \neq \{c^*\}$, then $\mathcal{W}(\pi_C^N) = \mathcal{W}^{Co}(\pi_C^N) \setminus \{c^*\}$; in all other cases, $\mathcal{W}(\pi_C^N) = \mathcal{W}^{Co}(\pi_C^N)$. This rule satisfies Anonymity for a given candidate, Independence of unconcerned voters, Independence of Pareto dominated candidates, Compromise, but violates Neutrality.

The logical independence of the properties for \mathcal{W}^{LD} and \mathcal{W}^{LA} are left to the reader.