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A Note on Efficiency in a Unifying Model of Strategic Network Formation*

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Abstract

The main point of this note is to provide a simpler and shorter proof of the result on efficiency in Olaizola and Valenciano (2017a) based on a result on efficiency of weighted networks in Olaizola and Valenciano (2017b). Additionally, this shorter proof allows to refine the result by establishing new conclusions for the zero-measure boundaries separating the different regions of values of the parameters where different structures were proved to be the only efficient ones.

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1 The result

The main point of this note is to provide a simpler and shorter proof of the result about efficiency in Olaizola and Valenciano (2017a) (in what follows "O&V'17a") based on a result on efficiency of weighted networks in Olaizola and Valenciano (2017b) (in what follows "O&V'17b"). In addition to this, the result is refined by establishing new conclusions for the zero-measure boundaries separating the different regions of values of the parameters where different structures are proved to be the only efficient ones, which are ignored in O&V'17a. We briefly review the model in O&V'17a, and the result about efficiency in O&V'17b. In order to minimize preliminaries and to avoid redundancies, the reader is referred to these two papers for further details and references therein.

1. *The unifying model*

O&V'17a studies a unification of two seminal connections models by Jackson and Wolinsky (1996) and Bala and Goyal (2000), eliminating the dichotomy of unilateral vs. bilateral formation of links. In the bridging model a link can be created by only one player (a "weak" link) by investing $c > 0$ in it. Then flow occurs in both directions with some degree of decay $\alpha \geq 0$. However, when a link is supported by *both* players by investing c each (a "strong" link), the degree of decay is δ $(1 > \delta \geq \alpha \geq 0)$. When the decay in weak links is maximal (i.e. $\alpha = 0$) we have Jackson and Wolinsky's connections model, where only strong links work, whereas when flow in weak links is as good as in strong links (i.e. $\alpha = \delta$) we have Bala and Goyal's two-way flow model, where strong links are inefficient.

Let N be a set of nodes and let graph g represent a strategy or investment profile, i.e. $g_{ij} = 1$ if and only if i invests c in link ij. For each pair of nodes $i \neq j$, let $\mathcal{P}_{ij}(g)$ denote the set of paths in g from j to i. For $p \in \mathcal{P}_{ij}(g)$, let $\ell(p)$ denote the length of p and $\omega(p)$ the number of weak links in p. Then i values information originating from j that arrives via p by $\delta^{\ell(p)-\omega(p)}\alpha^{\omega(p)}$. Thus, the *net value* or aggregate payoff of the network g is

$$v(g) = \sum_{i \in N} \sum_{j \in N(i;g)} \max_{p \in \mathcal{P}_{ij}(g)} \delta^{\ell(p)-\omega(p)}\alpha^{\omega(p)} - c(w + 2s), \qquad (1)$$

where $N(i;g)$ is the set of nodes directly or indirectly connected with i, and w and s are the numbers of weak and strong links in g. A strategy profile g is *efficient* if it maximizes the net value. Then the following result is proved:

Proposition 1 (*Olaizola and Valenciano,* 2017a) *If the net value is given by* (1) *with* $0 \leq \alpha < \delta < 1$, *then the only efficient networks, depending on the value of the parameters* (c, α, δ, n), *are the empty network, the complete network and the star network, in both cases with all the links of the same strength: either all weak or all strong, except perhaps in the boundaries separating the different regions where each of these structures is the only efficient one.*

In fact, Prop.1 in O&V'17a says more, because it specifies the complicated map corresponding to the different configurations of values of the four parameters for which each of the five architectures is the only efficient one. Nevertheless, Proposition 1 above, summarizes the most important part of the message: As in Jackson and Wolinsky (1996) and in Bala and Goyal (2000), the only nonempty efficient structures continue to be complete and star networks with all their links of the same strength, *except perhaps for a zero-measure set of values of the parameters*, but nothing is said about these boundaries, beyond that a tie of different efficient structures occurs. In spite of the richer variety of feasible structures in this model, possibly combining weak and strong links "which complicates considerably the proofs", no mixed structure appears to be efficient for any value of the parameters. The sentence between quotation marks is literally in O&V'17a, as the proof takes there 7 lemmas and 9 pages. Hence the interest of providing a shorter proof. Moreover, as the new proof shows, some other structures arise as efficient within this zero-measure region.

2. *Efficiency of weighted networks*

O&V'17b addresses the question of a planner whose objective is to form an efficient network in the sense of maximizing social welfare or aggregate utility of a set N of nodes. It is assumed that an investment of $c > 0$ in a link creates a link of strength $\lambda(c)$, where λ is a *link-formation technology*, i.e. a non decreasing map $\lambda : \mathbb{R}_+ \to [0,1)$ s.t. $\lambda(0) = 0$. Thus an investment vector $\mathbf{c} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ (where N_2 denotes the set of subsets of N of cardinality 2 and $\underline{ij} = \{i,j\}$), gives rise to a weighted network $g^{\mathbf{c}} = (g^{\mathbf{c}}_{\underline{ij}})_{\underline{ij} \in N_2}$, with $g^{\mathbf{c}}_{\underline{ij}} = \lambda(c_{\underline{ij}})$. It is assumed that a weighted network generates a net value $v(g^{\mathbf{c}})$ for each node in accordance with a set of assumptions (Assumptions 1-4) that we omit here, but, as is pointed out there, the unifying model in O&V'17a satisfies them all. Then the following result is established in O&V'17b:

Proposition 2 (*Olaizola and Valenciano,* 2017b) *Under Assumptions* 1-4*, for any link-formation technology* λ*, any network is dominated either by the empty network or by a connected dominant nested split graph* (*DNSG*) *network.*

Definition 1 *A nested split graph* (*NSG*) *is an undirected (weighted or not) graph such that*

$$\left|N^d(i;g)\right| \leq \left|N^d(j;g)\right| \Rightarrow N^d(i;g) \subseteq N^d(j;g) \cup \{j\}. \tag{2}$$

In other words, the adjacency matrix of g is a symmetric matrix such that, for a certain renumbering of the nodes, each row consists of a sequence of non-zero entries (apart from those in the main diagonal) followed by zeros, and the number of nonzero entries in each row is not greater than in the preceding row. An NSG-network is *dominant* if in addition satisfies the following definition in terms of the triangular matrix $T(g)$ above the main diagonal of 0-entries in adjacency matrix, i.e. $T(g) := (g_{ij})_{i<j}$:[1]

[1] In Olaizola and Valenciano (2017b) these networks are called *strongly* nested split graph (SNSG), but we find it more appropriate the term *dominant*.

Definition 2 (*Olaizola and Valenciano,* 2017*b*) *A dominant nested split graph* (*DNSG*) *network is a weighted NSG-network g such that, for a renumbering of the nodes satisfying* (*2*)*, in $T(g)$* : (*i*) *each row consists of a non-decreasing sequence of positive entries followed by zeros;* (*ii*) *all positive entries in the first row are greater than or equal to any other entries; and* (*iii*) *from the second row downwards, non-zero entries in the same column form a non-decreasing sequence.*

2 A shorter proof

Now we proceed to prove Proposition 1 as a corollary of Proposition 2. Note that the question of efficiency in the unifying model is equivalent to that of a planner looking for it by investing according to the following technology implicit in the model:

$$\lambda(x) = \begin{cases} \delta, \text{ if } x \geq 2c \\ \alpha, \text{ if } c \leq x < 2c \\ 0, \text{ if } x < c, \end{cases} \tag{3}$$

where $0 < \alpha < \delta < 1$. As mentioned before, it is immediate to check that the aggregate payoff or net value of a network designed by a planner using this technology satisfies all the assumptions under which Proposition 2 is established in O&V'17*b*. This yields the following first corollary:

Corollary 1 *If the net value of a network is given by* (1) *with $0 \leq \alpha < \delta < 1$, any network is dominated either by the empty network or by a connected DNSG-network.*[2]

Proof of Proposition 1:

In view of Corollary 1, it is enough to prove that any connected DNSG-network with positive net value which is *not* a network with all links of the same strength, either complete or an all-encompassing star, is strictly dominated by a network of this type, except perhaps within a zero-measure region of values of the of the parameters. In fact, it is immediate to check that in the region where $c \leq \delta - \delta^2$ and $c = 2(\delta - \alpha)$, i.e. the boundary separating the regions where the complete of strong links and the complete of weak links are the only efficient[3], *all* complete networks have the same value and are efficient. Assume that g is a connected DNSG-network s.t. $v(g) > 0$. Note that for technology (3) there can only be two types of links in g: strong links of strength δ and weak ones of strength α, and we can assume w.l.o.g. that they are invested in $2c$ and c respectively. Assume that the nodes are ordered so that conditions in Definitions 1 and 2 hold. Then the strongest link in g is $g_{\underline{1n}}$, and there are two cases:

Case 1: If $g_{\underline{1n}}$ is a *weak* link, i.e. $g_{\underline{1n}} = \alpha$, then *all* links in g must be weak, given its DNSG structure. In that case, g is dominated by a star or a complete network of weak links, strictly if it not one of them[4].

[2] A network g dominates (strictly) another g' if $v(g) \geq v(g')$ ($v(g) > v(g')$).

[3] See Figures 3, 4 and 5 in O&V'17*a*.

[4] The brief proof is identical to that of the result on efficiency in Jackson and Wolinsky (1996).

Case 2: If $g_{\underline{1n}}$ is a *strong* link, i.e. $g_{\underline{1n}} = \delta$, two cases are possible:

Case 2.1: *All links of the central star are strong*, i.e. $g_{\underline{1i}} = \delta$ for all $i = 2, ...n$. Assume g is *not* a star, i.e. $g_{\underline{ij}} \neq 0$, for some $i, j \neq 1$, nor the complete of strong links. If $\max\{2\delta - 2c, 2\alpha - c\} \leq 2\delta^2$ (eliminating a direct link, weak or strong, between two spoke nodes of the central star increases the net value), then g is strictly dominated by the all-encompassing star of strong links. Otherwise, if $\max\{2\delta - 2c, 2\alpha - c\} > 2\delta^2$, g is strictly dominated either by the complete network of strong links (if $2\delta - 2c > 2\alpha - c$) or by the complete of weak links (if $2\delta - 2c < 2\alpha - c$), or weakly by both if $2\delta - 2c = 2\alpha - c$.

Case 2.2: *The central star contains weak and strong links.* In this case,

$$g_{\underline{12}} = g_{\underline{13}} = ... = g_{\underline{1k}} = \alpha, \quad \text{and} \quad g_{\underline{1k+1}} = g_{\underline{1k+2}} = ... = g_{\underline{1n}} = \delta,$$

for some $1 < k < n$. If there are no more links, Lemma 5 in O&V'17a shows that then g is strictly dominated by an all encompassing star either with all links weak or all links strong. If any other links do exist, given the structure of DNSG-networks, they must be weak. Assume that $g_{\underline{ij}} = \alpha$, for some i and j s.t. $k + 1 \leq i, j \leq n$. Then, if $2\alpha - c > 2\delta^2$, by connecting all spoke nodes of the central star with weak links would yield a complete network of weak and strong links that strictly dominates g unless g contains all such links, which in turn, a similar discussion to that of case 2.1 one leads to the conclusion that it would be strictly dominated by a complete network with only one type of links, unless all complete networks yield the same value (i.e. $c = 2(\delta - \alpha)$). Otherwise, if $2\alpha - c \leq 2\delta^2$, such a link is superfluous, and g is dominated by the DNSG that results from deleting all such links.

Assume then in what follows that

$$c \geq 2\alpha - 2\delta^2 \tag{4}$$

and $g_{\underline{ij}} = 0$, for all i, j, s.t. $k + 1 \leq i, j \leq n$, i.e. *the submatrix of the last* $n - k$ *rows and* $n - k$ *columns consists of* 0's. There are two cases to be discussed:

Case 2.2.1: Assume that $g_{\underline{ij}} = \alpha$, for some i, j s.t. $2 \leq i \leq k$ and $k+1 \leq j \leq n$. If link $\underline{ij}$ is superfluous, the elimination of all such links yields a network that dominates g of the type considered in Case 2.2.2 discussed later. Otherwise, i.e. if a weak link improves the net value of the connection of two nodes through two links, one weak and another strong, i.e. $2\alpha - c > 2\delta\alpha$, then $2\alpha - c > 2\alpha^2$, i.e. a weak link also improves the net value of the connection of two nodes through two weak links. This leads to the conclusion that g is dominated by the DNSG-network whose adjacency matrix is of the form represented in Fig.1-(a):

$$
\begin{array}{c|ccccccc}
 & 1 & 2 & .. & k & k+1 & \dots & n \\
1 & 0 & \alpha & .. & \alpha & \delta & .. & \delta \\
2 & \alpha & 0 & \alpha & \alpha & \alpha & \alpha & \alpha \\
. & \vdots & \cdot & 0 & \alpha & \alpha & \alpha & \alpha \\
k & \alpha & \alpha & \cdot & 0 & \alpha & \alpha & \alpha \\
k+1 & \delta & \alpha & \cdot & \alpha & 0 & .. & 0 \\
. & \vdots & \cdot & \cdot & \alpha & 0 & \ddots & 0 \\
n & \delta & \alpha & \cdot & \alpha & 0 & .. & 0
\end{array}
\qquad
\begin{array}{c|ccccccc}
 & 1 & 2 & .. & k & k+1 & \dots & n \\
1 & 0 & \alpha & \dots & \alpha & \delta & \dots & \delta \\
2 & \alpha & 0 & \dots & \alpha & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & 0 & 0 & 0 \\
k & \alpha & \overline{\alpha} & \cdot & 0 & 0 & 0 & 0 \\
k+1 & \delta & 0 & \cdot & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \cdot & \cdot & \cdot & \cdot & \ddots & 0 \\
n & \delta & 0 & \dots & 0 & 0 & \dots & 0
\end{array}
$$

(a) (b)

Figure 1.

Before proceeding, we consider the second case.

Case 2.2.2: Assume that $g_{\underline{ij}} = \alpha$, for some i and j s.t. $2 \leq i, j \leq k$, while $g_{\underline{ij}} = 0$, whenever i or j is greater than k. If $\underline{ij}$ is superfluous, by eliminating all such links a dominant mixed star is obtained which is dominated by an all-encompassing star of either strong links or weak links (Lemma 5 in O&V'17a). Otherwise, this means that paying for a weak link improves the net value of the connection of two nodes through two weak links, i.e. $2\alpha - c > 2\alpha^2$, i.e.

$$c < 2\left(\alpha - \alpha^2\right). \tag{5}$$

This leads to the conclusion that g is dominated by the network whose adjacency matrix is of the form represented in Fig.1-(b) and (5) holds.

In sum, there only remains to be shown that the two types of DNSG-network represented in Fig.1 are strictly dominated. These two types of DNSG-network are exactly the "hybrid" networks introduced in the proof of lemmas 6 and 7 in O&V'17a: with the notation used there, g^*_{n-k} is represented in Fig.1-(b) and g^{**}_{n-k} in Fig.1-(a). More precisely, we have the following result that concludes the proof.

Lemma 1 *Any network of type g^*_{n-k} or type g^{**}_{n-k} is strictly dominated either by the complete network of weak links or by the all-encompassing star of strong links, except when $c = 2(\delta - \alpha) = \delta - \delta^2$.*

Proof. First note that in both cases (4) must hold, and if $k = 0$ then g^*_n and g^{**}_n are all-encompassing stars of strong links, while if $k = n$ then g^*_0 and g^{**}_0 are the complete network of weak links. Consider first a structure of type g^*_{n-k} for some k $(0 < k < n)$. Then the increase of value when passing from g^*_{n-k} to g^*_{n-k+1} and to g^*_{n-k-1} are[5]:

$$
\begin{aligned}
v(g^*_{n-k+1}) - v(g^*_{n-k}) &= 2\delta - 2\alpha + 2(k-2)(\alpha\delta - \alpha) + 2(n-k)(\delta^2 - \alpha\delta) + (k-3)c, \\
v(g^*_{n-k-1}) - v(g^*_{n-k}) &= -(v(g^*_{n-k+1}) - v(g^*_{n-k})) + 2\delta^2 - 4\alpha\delta + 2\alpha - c.
\end{aligned}
$$

[5]Note that only the contribution of pairs containing node k in one case, and $k+1$ in the other, to the net value of the network change.

Therefore, if $v(g^*_{n-k+1}) - v(g^*_{n-k}) \leq 0$,

$$v(g^*_{n-k-1}) - v(g^*_{n-k}) \geq 2\alpha + 2\delta^2 - 4\alpha\delta - c > 2\alpha - 2\alpha^2 > 0,$$

because, as $\alpha < \delta$, $(\alpha - \delta)^2 > 0$, i.e. $\alpha^2 + \delta^2 - 2\alpha\delta > 0$, and (5). In sum,

$$v(g^*_{n-k+1}) - v(g^*_{n-k}) \leq 0 \quad \Rightarrow \quad v(g^*_{n-k-1}) - v(g^*_{n-k}) > 0.$$

From here it follows easily that g^*_{n-k} is necessarily strictly dominated either by g^*_0 or by g^*_n.

Consider now a structure of type g^{**}_{n-k} for some k $(0 < k < n)$. Then the increase of value when passing from g^{**}_{n-k} to g^{**}_{n-k+1} and to g^{**}_{n-k-1} are:

$$\begin{aligned} v(g^{**}_{n-k+1}) - v(g^{**}_{n-k}) &= 2\delta - 2(n-k+1)\alpha + 2(n-k)\delta^2 + (n-k-1)c, \\ v(g^{**}_{n-k-1}) - v(g^{**}_{n-k}) &= -\left(2\delta - 2(n-k)\alpha + 2(n-k-1)\delta^2 + (n-k-2)c\right). \end{aligned}$$

Therefore,

$$v(g^{**}_{n-k+1}) - v(g^{**}_{n-k}) = -(v(g^{**}_{n-k-1}) - v(g^{**}_{n-k})) - 2\alpha + 2\delta^2 + c.$$

Now, if $-2\alpha + 2\delta^2 + c > 0$ (i.e. $c > 2\alpha - 2\delta^2$), it follows, as for g^*_{n-k}, that

$$v(g^{**}_{n-k+1}) - v(g^{**}_{n-k}) \leq 0 \quad \Rightarrow \quad v(g^{**}_{n-k-1}) - v(g^{**}_{n-k}) > 0,$$

from which it easily follows that g^{**}_{n-k} is strictly dominated either by g^{**}_0 or by g^{**}_n. Finally, if $c = 2\alpha - 2\delta^2$, then

$$v(g^{**}_{n-k+1}) - v(g^{**}_{n-k}) = -(v(g^{**}_{n-k-1}) - v(g^{**}_{n-k})).$$

In other words, in the sequence from g^{**}_0 to g^{**}_n the increase (decrease) of net value is the same for any two consecutive structures. From which, it also follows that g^{**}_{n-k} is strictly dominated either by, g^{**}_0 or by g^{**}_n, unless all these differences are 0. However, this can only occur if in addition $c = 2(\delta - \alpha) = \delta - \delta^2$, which corresponds to the intersection of the boundaries of the regions where the star of strong links, the complete network of strong links and the complete of weak links are the unique efficient network[6]. For this very particular configuration of values of the parameters all three structures and also all g^{**}_{n-k} are efficient and have the same net value. ■

In fact, in addition to proving Proposition 1, we have shown the following result that completes Proposition 1 by refining the conclusions about the boundaries:

Proposition 3 *If the net value is given by* (1) *with* $0 \leq \alpha < \delta < 1$*: (i) In the zero-measure region where* $c \leq \delta - \delta^2$ *and* $c = 2(\delta - \alpha)$ *all complete networks have the same value and are efficient. (ii) In the zero-measure region where* $c = \delta - \delta^2 = 2(\delta - \alpha)$ *in addition to all complete networks, all DNSG-networks of type* g^{**}_{n-k}*, including the star of strong links, have the same value and are efficient.*

[6] See Figures 3, 4 and 5 in O&V'17*a*.

3 Concluding comment

As said in the proof, the two types of DNSG-network represented in Fig.1 are the special networks introduced in the proof of Lemmas 6 and 7 in O&V'17a, where we called them "hybrid structures" because we were not aware of the existence nested split graph networks by then. We introduced them as terms of reference to be able to conclude the proof of the result about efficiency once there only remained to be explored the region of values of the parameters specified by conditions

$$\max\{2\delta - 2\alpha, 2\alpha - 2\delta^2\} < c < 2\alpha - 2\alpha^2.$$

It is remarkable that these structures appear naturally as the only possible efficient DNSG-networks in the final stages of the simpler proof provided here.

As mentioned in the first section, Prop. 1 in O&V'17a also establishes the precise boundaries of the five regions where each of the five different structures is the only efficient one. In fact, it is easy to derive these boundaries once settled Proposition 1. Moreover, Proposition 3 shows that, even if only for very special configurations of values of the parameters within these boundaries, other structures can be efficient, thus refining O&V'17a's conclusions.

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