# PLANCKS 2019 Problem Set

Dear contestants, Welcome to PLANCKS 2019

- The language used in the competition is English

- The contest consists of **10 exercises, each worth 10 points**. Subdivisions of points are indicated in the exercises.

- All exercises have to be handed in separately. For this purpose you have been given envelopes. Please mark each envelope with your team code and the exercise number. **Marking the envelope is necessary for an exercise to be scored**

- When a problem is unclear, a participant can ask, via the crew, for a clarification. If the response is relevant to all teams, the jury will provide this information to the other teams.

- You are allowed to use a dictionary: English to your native language.

- You are allowed to use a non-programmable, not-graph calculator (But scientific is okay).

- No books or other sources, except for this exercise booklet and a dictionary, are to be consulted during the competition.

- The organisation has the right to disqualify teams for misbehaviour or breaking the rules.

- The use of hardware (including phones, tablets etc.) is not approved, except for watches and medical equipment. Please leave your phones in an envelope.

- In situations to which no rule applies, the organisation decides. We wish you all the very best.

  May the best physics team win!

1   Mono- and Multilayer adsorption

Monolayer adsorption (Langmuir isotherm)   Multilayer adsorption (BET isotherm)
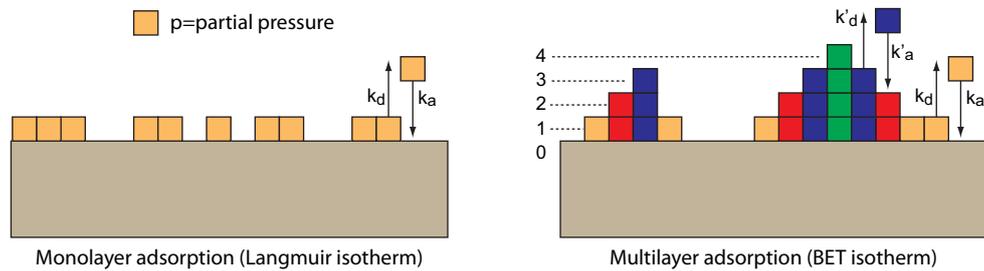
Adsorption of molecules from the gas phase and onto a solid surface represents an important class of problems in surface physics. To model the process, some assumptions are usually made: (1) The surface has a finite number of identical adsorption sites into which molecules can adsorb. (2) The adsorbed molecules do not interact. (3) Binding is reversible.

First we consider adsorption to form a monolayer on the surface. That is, the formation of multilayers is not allowed. Here N_0 is the number of empty surface sites and N_1 is the number of occupied sites. The partial pressure of the adsorbate in the gas phase is p. The rate constant for adsorption is k_a and the rate constant for desorption is k_d. The equilibrium constant is $K = k_a/k_d$.

The rate of adsorption is: $r_a = k_a p N_0$
The rate of desorption is: $r_d = k_d N_1$

(a, 2 points) Assume that adsorption has reached equilibrium. **Derive an expression for the monolayer surface coverage Θ in terms of K and p**. Note that Θ is defined as the fraction of surface sites occupied by adsorbates:

$$\Theta = \frac{N_\text{ad}}{N_\text{sites}}$$

-where N_ad is the number of surface sites having adsorbates and N_sites is the total number of surface sites.

(b, 2 points) What is Θ in the limiting situations of high and low pressures p? What is the characteristic pressure $p_{1/2}$ where half of the surface sites are occupied?

Next we consider the situation where adsorbing molecules can form multilayers on the surface[1]. We define N_i as the number of surface sites having exactly i adsorbates. The same assumptions (1-3) as above are used. In addition we assume that: (4) The gas molecules only interact with the top layer of the surface. (5) There is only two types of interactions: (I) adsorbate-surface and (II) free adsorbate-adsorbate interactions. We therefore define two new rate constants k_a' and k_d' to describe the adsorbate-adsorbate interactions. The equilibrium constant for adsorbate-adsorbate interactions is $K' = k_a'/k_d'$. In order to simplify calculations we define $c = \frac{K}{K'}$.

(c, 2 points) Using a recursive argument show that N_i can be written in terms of N_0. Show specifically that for $i > 1$:

$$N_i = c(K'p)^i N_0 = cz^i N_0$$

-where we have defined $z = K'p$.

(d, 2 points) Next we are interested in finding the surface coverage Θ for multilayer adsorption, where $\Theta = \frac{N_{\text{ad}}}{N_{\text{sites}}}$, with the multlayer definition of N_{ad}. For multilayer adsorption the total number of adsorbed molecules is given as:

$$N_{\text{ad}} = N_1 + 2N_2 + 3N_3... = \sum_{i=1}^{\infty} iN_i$$

The total number of surface sites is the sum of sites having all possible numbers of layers (including empty sites):

$$N_{\text{sites}} = N_0 + N_1 + N_2 + N_3... = \sum_{i=0}^{\infty} N_i$$

Find expressions for N_{ad} and N_{sites} for multilayer adsorption.
Determine the coverage Θ for multilayer adsorption.

(e, 2 points) Give a physical interpretation of the quantities K' and z.

2 Quantum Optics

The quantum behaviour of light can be observed in extremely simple optics experiments, with probably the simplest being a single *beamsplitter*. An important consequence of the quantization of light is that single quanta of light - photons - can not simply be split in half in a linear beamsplitter. Quantum mechanics instead describes the probabilities of each individual photon emerging from one of the beamsplitter outputs. This in itself already leads to observable results revealing the quantum statistics of photons and allows us to understand the classical Mach-Zehnder interferometer in terms of single photons.

Exercise 1:   The classical beamsplitter

A beamsplitter is simply a partially reflective mirror, where each of the two input ports is broken into a transmitted and a reflected part, as shown in the figure. We consider a lossless beamsplitter with classical inputs E_1 and E_2. The outputs are then given by

$$E_3 = RE_1 + TE_2, \; E_4 = TE_1 + RE_2, \quad (1)$$

where E_i are the (complex) amplitudes of the input and output light fields, R is the reflection coefficient, and T the transmission coefficient. Energy conservation requires that $|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$. Note, that here we consider the intensities, while the reflection and transmission coefficients are defined relative to the field amplitudes.
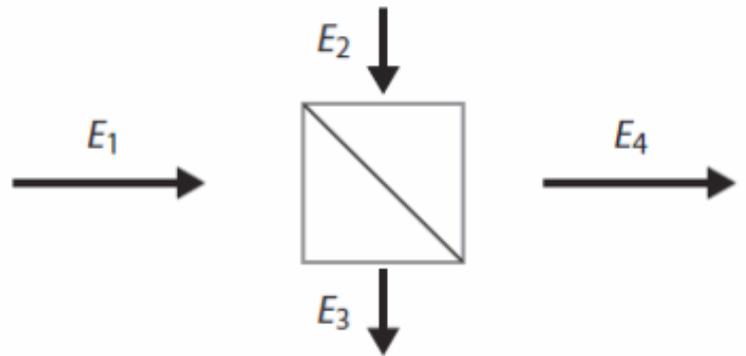
a) *(1 point)* Show that energy conservation for the case of only one input light field (i.e. $|E_2| = 0$) leads to $|R|^2 + |T|^2 = 1$.

b) *(1 point)* Show that the case of equal field strengths on both inputs i.e. $E_1 = E_2 = E$) leads to the condition $RT^* + TR^* = 0$.

b) *(1 point)* Show that the above condition leads to an additional phase shift of $\pi/2$ between the fields of the transmitted and reflected components.

Exercise 2:   The QM beamsplitter in braket notation

We now consider the case where the input into the beamsplitter are single photons. Two photon sources, A and B, emit photons into the respective input modes of the beamsplitter, and two detectors, C and D detect photons in the respective output modes. Since the photons are non-interacting and independent of each other, we need to assign each photon j an invididual state vector $|x\rangle_j$ where their state x expresses
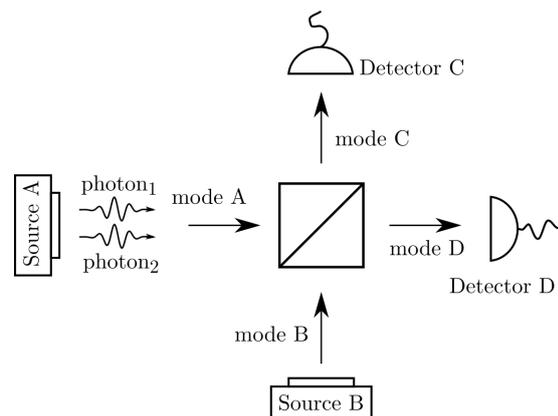
the mode X it is located in. The overall state
for n incoming photons in mode A would thus
be

$$|\psi^{\text{in}}\rangle = |a\rangle_1 |a\rangle_2 \cdots |a\rangle_n, \tag{1}$$

while a state with photon 1 in mode A and photon 2 in mode B would be

$$|\psi^{\text{in}}\rangle = |a\rangle_1 |b\rangle_2, \tag{2}$$

which is not identical to photon 1 in mode B and photon 2 in mode A

$$|\psi^{\text{in}}\rangle = |b\rangle_1 |a\rangle_2. \tag{3}$$

The beam splitter either transmits or reflects each single photon, with the reflection and transmission coefficients from the previous exercise determining the probability of each outcome. For the case of a loss-free 50/50 beam splitter the output for single photons emitted in modes A and B, described by $|a\rangle$ and $|b\rangle$ respectively, is

$$|a\rangle \xrightarrow[BS]{} \frac{1}{\sqrt{2}}(|c\rangle + i\,|d\rangle), \tag{4}$$

$$|b\rangle \xrightarrow[BS]{} \frac{1}{\sqrt{2}}(i\,|c\rangle + |d\rangle). \tag{5}$$

The factors of $i = \sqrt{-1}$ originates from the phase shift induced on reflected photons which you proved in the last exercise.

a) *(2 points)* For the following input states, calculate the output state and the probabilities of specific detection events:

(a) A single photon in mode A: $|\psi^{\text{in}}\rangle = |a\rangle$. What is the probability of detecting the photon on counter C? And on counter D?

(b) Two photons in mode A: $|\psi^{\text{in}}\rangle = |a\rangle_1 |a\rangle_2$. What is the probability of detecting both photons on counter C? On counter D? What is the probability of detecting one photon on each counter?

b) *(2 points)* Consider the following input states with one photon in each input mode:

(a) two distinguishable photons $|\psi^{\text{in}}\rangle = |a\rangle_1 |b\rangle_2$,

(b) two indistinguishable bosonic photons $|\psi^{\text{in}}\rangle = \frac{1}{\sqrt{2}}(|a\rangle_1 |b\rangle_2 + |b\rangle_1 |a\rangle_2)$,

(c) a superpositon of two 'fermionic photons' $|\psi^{\text{in}}\rangle = \frac{1}{\sqrt{2}}(|a\rangle_1 |b\rangle_2 - |b\rangle_1 |a\rangle_2)$.

Calculate the output state for each input state and determine the probabilities of detecting both photons on detector C, both on detector D, and one photon on each detector.

Exercise 3: Hong-Ou-Mandel experiment

The experiment you discussed in the previous exercise was first performed by Hong, Ou and Mandel in 1987. As shown in the figure, they let two photons, one in each input mode, on the beamsplitter.
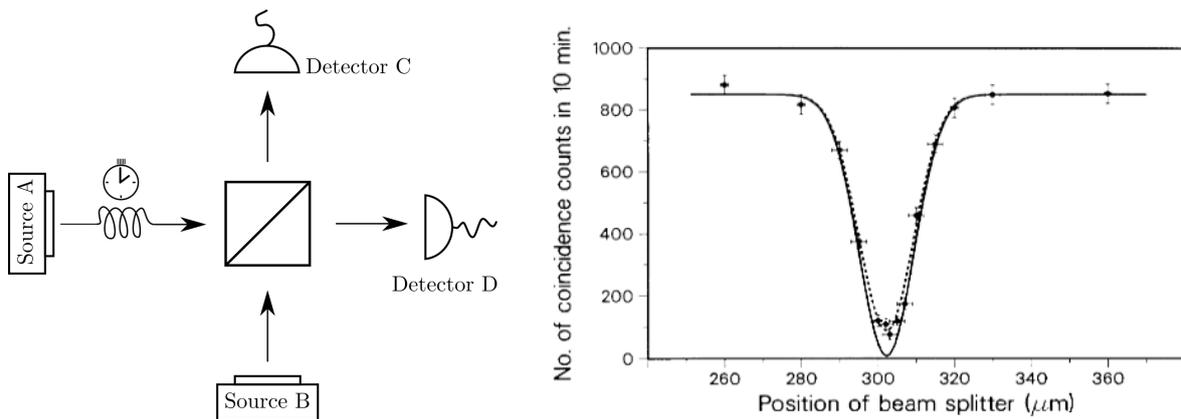
Figure 1: Left: Schematic of the Hong-Ou-Mandel experiment. Right: Plot of the first observation of the Hong-Ou-Mandel dip (C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. 59, 2044, 1987).

The experiment had an additional parameter, namely the distance between source A and the beam-splitter. This allowed to change the relative arrival time of the two photons on the beamsplitter. In their data original plot, shown in the figure, this corresponds to what is shown on the x-axis. A beamsplitter position of slightly more than $300\mu m$ meant that the photons arrived exactly at the same time, while a change of this position made the photon from source A arrive earlier or later than the one from source B. With this setup, they could measure the probability to detect a *coincidence* between the detectors as a function of the delay length (equivalent to delay time). Coincidence means that one photon each is detected on each detector simultaneously.

As you can see in the figure the results of this photon coincidence experiment shows a strong dip when the photons arrive within a certain time window. This is now known as the Hong-Ou-Mandel dip.

a) *(1 point)* Consider the experimental result shown in the figure. What does the observed dip in the coincidences tell about the quantum statistics of photons when you compare it to your results from exercises 1 b) and 2 b)?

b) *(1 point)* The Hong-Ou-Mandel experiment is nowadays used as an important tool in quantum optics, e.g. to determine the quality of single-photon emitters. In such experiments, only a single source is used, with the first photon sent to input A being delayed, e.g. in a long optical fiber, so that it coincides with the second photon (from the same source) sent to input B. Again, this relative delay is varied as in the original experiment, and a dip is measured as function of the delay time. What do you think decides the width (in time - or distance as in the plot shown here) of the dip? Considering your results from the previous exercise what do you think can be learned by the depth of the dip?

c) *(1 point)* How does the Hong-Ou-Mandel dip fit together with the statement that photons are non-interacting particles?

3 Radical pair spin chemistry

Consider a "radical pair" which consist of two unpaired electronic spins, where one of the electrons is coupled to a spin-1/2 nucleus through one hyperfine interaction tensor A. The radical pair is subject to an external magnetic field $\vec{B}$. The overall spin state of the two unpaired electronic spins can be either singlet (S) or triplet (T).The Hamiltonian for the system is given by

$$\hat{H} = \mu_B g(\vec{B} \cdot \vec{S}_1) + \mu_B g(\vec{B} \cdot \vec{S}_2) + \mu_B(\vec{S}_1 \cdot \hat{A} \cdot \vec{I}) \tag{1}$$

when the possible dipole-dipole and exchange interactions in the spin-system are neglected. $\vec{I} = (I_x, I_y, I_z)_i$ is the spin operator of the nucleus, $\vec{S}_{1,2} = (S_x, S_y, S_z)$ are the electron spin operator and $\hat{A}$ is the hyperfine interaction tensor.

(**Q1:**) *(1 points)* What are the possible values of the total spin?

(**Q2:**) *(3 points)* Define all possible basis wavefunctions ψ_i, $i = 1, 2, ..., 8$ that are eigenfunctions of the total spin, to describe the quantum states of the radical pair.

(**Q3:**) *(2 points)* State the conditions at which a transition between a singlet and triplet state of the radical pair is possible.

(**Q4:**) *(1 points)* Check numerically if this condition is satisfied for the anisotropic hyperfine tensor, where the $A_{zz} = 16\,G$ is the only non-zero component. The external magnetic field strength to be 0.5 G.

(**Q5:**) *(3 points)* Now consider the radical pair "prepared" initially in the singlet state, estimate the characteristic time needed for it to be converted into one of the triplet states.

4 Relativistic Orbit

It is well known that planets move in elliptical orbits around the sun and the derivation of the orbit equation is a standard exercise in classical mechanics. However, if the effects of special relativity are taken into account, the orbit is a rotating ellipse of the form

$$\frac{1}{r} = \frac{1}{r_0} \left\{ 1 + \varepsilon \cos[\alpha(\theta - \theta_0)] \right\} \ ,$$

where $\alpha = 1$ corresponds to the classical result.

- *(1 point)* Show that for small velocities the relativistic kinetic energy (perhaps up to a constant) of a body of mass m can be approximated by $\frac{-mc^2}{\gamma}$.

- *(1 point)* Using the result from the previous point, write down the relativistic Lagrangian for the sun-planet system.

- *(5 point)* Derive the above equation for the orbit and express α and r_0 in terms of fundamental constants of the trajectory (such as energy, angular momentum, etc.).

- *(3 point)* Given that the mean radius of the orbit of Mercury is 58×10^6 km and that its orbital period is 88 days, calculate the shift in the angular coordinate of the perihelion (orbit point nearest to the sun) over a century.
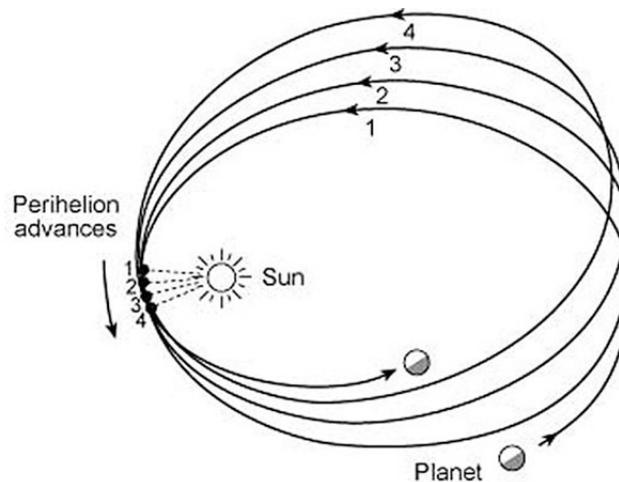
5 Polymers and Rubber

In the following exercise, we will introduce the statistical physics of polymer molecules and use it to make predictions for the force-extension relation for a rubber band. A polymer molecule is a linear chain of monomers. Rubber is a material where a large number of polymers are chemically cross-linked into one large molecule.

Exercise 1 Pulling a single polymer molecule

We can model a polymer molecule as a random walk consisting of N steps each of length b. Lets say one end of the polymer starts at the origin of our coordinate system, then the probability for finding the other end of the polymer at $\mathbf{R} = (x, y, z)$ is given by the Gaussian distribution

$$P(x, y, z; N) = \left(\frac{3}{2\pi N b^2}\right)^{\frac{3}{2}} \exp\left(-\frac{3(x^2 + y^2 + z^2)}{2N b^2}\right)$$

1. *(1 point)* Derive expressions for the 1D and 3D mean-square extension, i.e. the moments $\langle x^2 \rangle$ and $\langle R^2 \rangle = \langle x^2 + y^2 + z^2 \rangle$

2. *(1 point)* Calculate the average spatial size (the root-mean-square end-to-end distance) and compare it to the contour length of the polymer $L = bN$. The latter is the maximal length we can pull the polymer. For polyisoprene (synthetic rubber) typical numbers are $N = 20000$ and $b = 8\mathring{A}$.

Imagine we perform an AFM experiment, where we have fixed one end of our polymer on a substrate and the other end of the polymer on the AFM cantilever, such that we control the end-to-end vector $\mathbf{R}$. Simutaneously, we can measure the force $F(\mathbf{R})$ the polymer generates on the cantilever. In a statistically physical sense the macrostate is defined by $\mathbf{R}$, while the number of corresponding microstates is $\Omega(\mathbf{R}) \propto P(x, y, z)$. Because the probability (up to a normalization) is the number of states, that exactly has the specified end-to-end distance.

3. *(1 point)* Derive an expression for the (Helmholtz) free energy $A(\mathbf{R})$. Neglect the internal energy U, as well as all terms independent of $\mathbf{R}$. Finally, show the polymer force is entropic and is given by

$$F(\mathbf{R}) = -\frac{3kT}{N b^2}\mathbf{R}$$

Exercise 2 Deformation free energy density

We assume the cross-linking process takes place in the undeformed state. To model it, we assume all the strands in the network are formed with end-to-end distances $\mathbf{R} = (x, y, z)$ sampled from the Gaussian distribution $P(x, y, z; N_s)$, where for simplicity we assume all strands in the network has the number of steps N_s.

When we subsequently deform the material, we assume each strand experience a homogeneous affine deformation such that the end-to-end vector (x, y, z) is transformed to $(\lambda_x x, \lambda_y y, \lambda_z z)$. But the internal monomers in the polymer remains free to move. Furthermore, since rubber is incompressible all deformations occur at constant volume. Assuming we arbitrarily pull in the x direction, then the elogation ratios are simply given by $\lambda_x = \lambda$, and $\lambda_y = \lambda_z = \lambda^{-\frac{1}{2}}$

1. *(1 point)* Calculate the mean-square end-to-end distance of a deformed strand $\langle \lambda_x^2 x^2 + \lambda_y^2 y^2 + \lambda_z^2 z^2 \rangle$ using the results of the previous exercise. (Hint: averages in the deformed state are taken with respect to the probability $P(x, y, z; N_s)$, because we cross-linked the network in the undeformed state)

2. *(1 point)* Use the results of exercise 1 to calculate the average free energy $\langle A(\lambda) \rangle$ of a single deformed strand, and show that the Helmholtz free energy density change $\Delta a(\lambda)$ relative to the undeformed state is given by

$$\Delta a(\lambda) \equiv \rho_s \left(\langle A(\lambda) \rangle - \langle A(1) \rangle \right) = \frac{\rho_s k_B T}{2} \left(\lambda^2 + 2\lambda^{-1} - 3 \right),$$

where ρ_s is the number density of network strands, and the shear modulus of the material is defined as $G = \rho_s k_B T$.

3. *(1 point)* Calculate the strand number density ρ_s and the shear modulus. For poly-isoprene: bulk density $\rho = 0.830 g/cm^3$, the mass of a single step $M_0 = 113 g/mol$, and each strand consists of $N_s = 44$ steps. We assume the material is at room temperature.

Exercise 3 Force-extension for a rubber band

The force $F(\lambda)$ we required to pull the rubber band to maintain a length $L_x(\lambda)$ or equivalently extension λ is

$$F(\lambda) = \frac{d \left(V \Delta a(\lambda) \right)}{d L_x(\lambda)}$$

where V denotes the volume of the rubber band, and the length is $L_x(\lambda) = L_{x1} \lambda$. L_{x1} is the undeformed length. Our external force is equal and opposite to the force generated by the material, hence the sign in this expression.

1. *(1 point)* Make a sketch of the force-extension $(L_x(\lambda), F(\lambda))$ relation for a rubber band of undeformed dimensions $L_{x1}, L_{y1}, L_{z1} = 10cm, 0.3cm, 0.3cm$. Use the value for the shear modulus in the preceeding exercise, or failing that use the estimate $G = 0.5 MPa$.

2. *(1 point)* Derive approximate expressions for force-extension relationship valid in the small and large deformation limits, respectively. (NB. The strain ϵ is defined as $\lambda = 1 + \epsilon$ and we assume $\epsilon \ll 1$ in the small deformation limit)

3. *(1 point)* Assume we suspend a mass $m = 1kg$ from the end of the rubber band. What is the corresponding equilibrium length L_x? (NB. assume we are in the large deformation limit.) Now imagine that you instantaneously heat the rubber band to $T = 500K$ by running the flame of a lighter along it. What is the corresponding equilibrium length?

4. *(1 point)* If we pull a rubber band too much it snaps. The end-to-end distance of a polymer (strand) can not exceed the contour length of the polymer (strands), since we would have pulled it to a straight line. Use the results of the preceeding exercises to estimate the breaking point λ_c where the (root mean square) end-to-end distance match 50% of the contour length of a strand. (NB. assume we are in the large deformation limit).

The shear modulus is proportional to temperature because rubber elasticity is an entropic effect, which makes rubber materials very different from most other materials. This was observed by James Gough in 1802 and was later studied by Joule. The effect is now known as the Joule-Gough effect.

 0

6   Topological phase transition in the 2D YX-model

In 2016 the Noble Prize in Physics was awarded to three British physicists John Kosterlitz, David Thouless and Duncan Halfdane for their work on topological phases of matter and topological phase transitions, which have had major impact on many areas of physics. In this exercise we will go through some simple considerations in a statistical mechanical setting to illuminate some characteristics of topological defect structure. Topological defects in physical systems emerges as point or string-like structures in the low-temperature "ordered" state of systems which possess continuous symmetry, e.g. ferro-magnets, nematic liquid crystals, quantum liquids and superconductors. The relevant topological defects depend on the symmetry and the dimension of space, but the simplest examples are found in 2D ferromagnets. The standard O(2) symmetric model of ferro-magnetism on a regular 2D lattice (XY-model) takes the form

$$\mathcal{H} = -\frac{J}{2}\sum_{\langle ij\rangle}\vec{S}_i\vec{S}_j \tag{2}$$

where each lattice site $i = 1, .., N$ is equipped with the variable $\vec{S}_i = (\cos(\theta_i), \sin(\theta_i)) \in S^1$. θ_i is the angle between $\vec{S}_i$ and the x-axis. $J > 0$ is the nearest neighbor coupling strength.

Problem 1: *2 points* Show that at low temperatures Eq.(2) can be approximated by a continuum model

$$\mathcal{H} \simeq \frac{Kz}{2}N + \frac{K}{2}\int_A d^2x\,\partial_\mu\theta(\vec{x})\partial_\mu\theta(\vec{x}) \tag{3}$$

where $\vec{x} = (x_1, x_2)$ and $\partial_\mu\theta = \frac{\partial\theta}{\partial x_\mu}$, $\mu = 1, 2$. A is the area, z is the lattice coordination number and $K = \frac{J}{2}c$, where c is a geometrical factor of order unity depending on the lattice employed. Furthermore, Eq.(3) has a short distance cut-off a which is the lattice spacing.

Problem 2: *2 points* Verify that the energetically most favorable configurations of the angle field $\theta(\vec{x})$ at low temperatures obey:
$$\partial_\mu\partial_\mu\theta(x) = \vec{\nabla}^2\theta(x) = 0 \tag{4}$$

As $\theta(\vec{x})$ obey the 2D Laplace equation, - it is a harmonic field. $\theta = $ constant is clearly a solution of Eq.(4), consistent with our expectation at $T = 0$. Since $\vec{S}$ is periodic we must in general expect that the accumulated angle θ along a closed curve C obey

$$\oint_C d\theta = \oint_C \partial_\mu\theta(\vec{x})dx_\mu = 2\pi q, \quad q \in Z \tag{5}$$

If $q = 0$, $\theta(\vec{x})$ is regular (analytic) inside the curve C. If $q \neq 0$ there is at least one (non-analytic) singular point of $\theta(\vec{x})$ inside C. q is called the topological index or strength, which is clearly an additive property. Let us assume that q is invariant for all paths encircling Origo $(0, 0)$. Consider a tiny circular curve of radius r around Origo, which in polar coordinates can expressed as $\vec{x}(r, \phi) = r\vec{e}_r$ where $\vec{e}_r = (\cos(\phi), \sin(\phi))$. In these coordinates we have $\vec{\nabla}\theta = \frac{1}{r}\frac{\partial\theta}{\partial\phi}\vec{e}_\phi + \frac{\partial\theta}{\partial r}\vec{e}_r$ where

$\vec{e}_\phi = (-\sin(\phi), \cos(\phi))$ and $\vec{\nabla}^2\theta(x) = \frac{\partial^2\theta}{dr^2} + \frac{1}{r}\frac{\partial\theta}{dr} + \frac{1}{r^2}\frac{\partial^2\theta}{d\phi^2}$.

Problem 3: *1 point* Show that

$$\theta(x_1, x_2) = \theta_0 + q\phi(x_1, x_2) = \theta_0 + q\tan^{-1}(x_2/x_1) \tag{6}$$

is a solution of Eq.(4) in $\mathbb{R}^2/0$ which satisfy Eq.(5).

The resulting *textures* of $\vec{S}$ is shown in Figure 2 for $\theta_0 = 0$ and $q = -1, 1, 2$.

Problem 4: *1 point* In Figure 3 is shown the texture associated with two point defects. What is the strength of the two defects?

The energy associated with these particle-like defect textures:

$$\mathcal{H} = \frac{K}{2}\int_a^R \int_0^{2\pi} rdrd\phi(\vec{\nabla}\theta)^2 = \pi q^2 K \ln(R/a) \tag{7}$$

where R represents the system size, i.e. $\pi R^2 = A$.

Problem 5: *1 point* Evaluate the self-energy of a defect of strength q. (i.e. show that eq. 7 holds) and explain why so $|q| = 1$ defects are dominating the system

The above considerations for a single point-defect can easily be generalized to a multi-defect system, where the defect solutions can be added and so can their strengths. An example is shown in Figure 4

Nothing is confining the defects to a particular place on the lattice, so they are free to move.

Problem 6: *1 point* Calculate the translational entropy of a single defect.

Now, consider a pair of defects of strength $+1$ and -1. The total topological strength of the system is thus zero and the two defects can annihilate. Similarly, such a pair can be created spontaneously with an energy cost Eq(7) for each defect.

Problem 7: *2 points* Obtain an expression for the free energy of such a pair of defects, and find the temperature above which defect pairs spawn spontaneously.

Kosterlitz and Thouless (1973) argued that above this temperature the spontaneous creation and proliferation of defects will lead to a complete disordering of the system and an infinite order phase transition

Not a problem: *0 points* What does it mean for a transition to be infinite order?
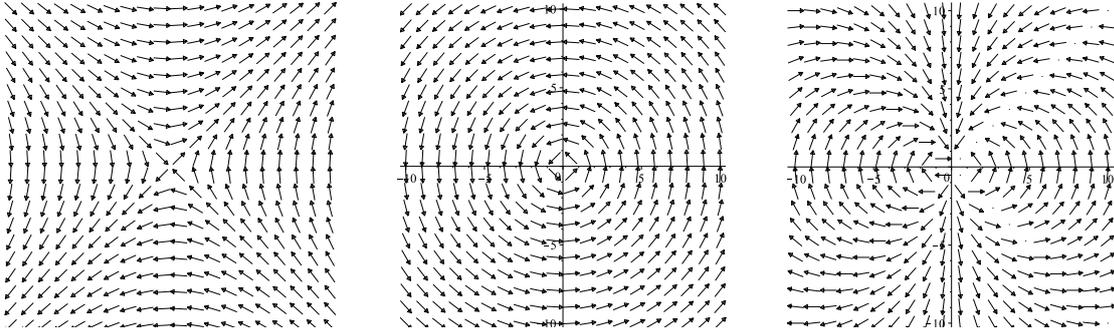
Figure 2: Examples of textures for defects with strength: $q = -1, 1, 2$ respectively
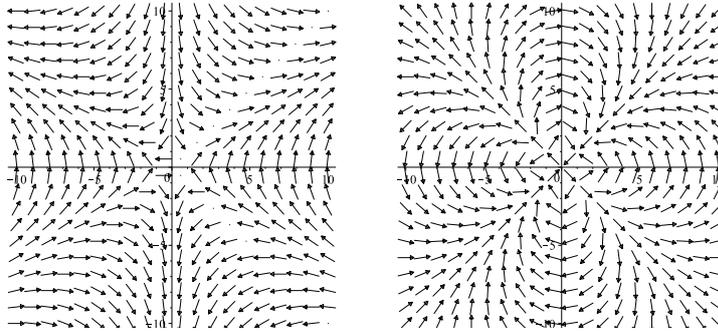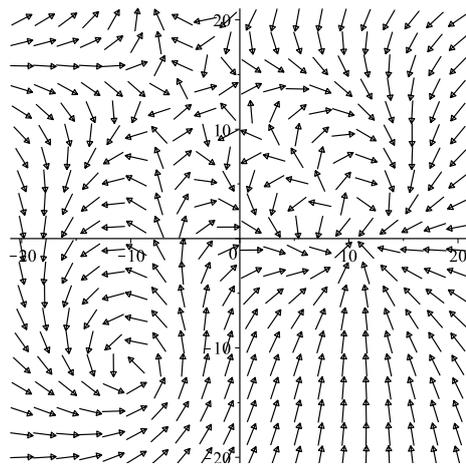

Figure 3: Textures for 2 more defects.

Figure 4: $(\cos(\phi(x_1, x_2)), \sin(\phi(x_1, x_2)))$ field for $\phi(x_1, x_2) = \frac{\pi}{2} + \tan^{-1}((x_2 + 12)/(x_1 + 10)) - 3\tan^{-1}((x_2 + 5)/(x_1 - 13)) + 2\tan^{-1}((x_2 - 1)/(x_1 - 10))$

7 Higgs Mechanism

A long range force like the electromagnetic force is mediated by massless gauge bosons. A force like the weak force is short range and its strength decreases exponentially with the distance. The mediating gauge bosons are massive. While the electromagnetic force is connected to the Coulomb potential, the weak force can be derived from the Yukawa potential

$$V^{\text{Yuk}}(r) = \frac{e^{-\frac{cM}{\hbar}r}}{4\pi r} \tag{8}$$

where M is the mass of the mediating gauge boson. The Yukawa potential is the solution of the Klein-Gordon equation for a scalar potential with a pointlike source at the origin.

The Higgs mechanism allows to introduce masses for gauge bosons without destroying the underlying gauge symmetry.

The mechanism can be illustrated by looking at the example of a superconductor. An electromagnetic field entering a superconductor is exponentially suppressed. This effect is the so-called Meißner–Ochsenfeld effect.

The relation between a magnetic field $\vec{B}$ and the superconducting current density $\vec{j}_s$ is given by the London equations:

$$\nabla \times \vec{j}_s = -\frac{(2e)^2 n_c}{m_c c^2}\vec{B} \tag{9}$$

where $n_c = \frac{1}{2}n_e$ is the number density of the Cooper pairs, n_e the number density of the electrons associated to the superconductivity. The mass of a Cooper pair is denoted by $m_c = 2m_e$ and m_e is the mass of the electron.

a) *(2 points)* Assume a static case with

$$\frac{\partial \vec{E}}{\partial t} = 0 \tag{10}$$

and use the Maxwell's equations (in SI units) to derive

$$\nabla^2 \vec{B} = +\underbrace{\frac{\mu_0 e^2 n_e}{m_e c^2}}_{=\frac{1}{\lambda^2}}\vec{B} \tag{11}$$

where λ is the penetration depth.

b) *(2 points)* Assume a magnetic field in z-direction with its absolute value depending on x, $\vec{B} = B(x)\vec{e}_z$, and find a solution to the differential equation resulting from Eq. (11) with the boundary condition that $\vec{B}$ is vanishing for $x \to \infty$.

c) *(1 point)* Compare the given Yukawa potential and the found solution for the $\vec{B}$ field and guess a relation between the penetration length λ and the mass M.

d) *(3 points)* Show that the vector potential $\vec{A}$ with $\vec{B} = \nabla \times \vec{A}$ in Coulomb gauge ($\nabla \cdot \vec{A} = 0$) fulfills

$$-\nabla^2 \vec{A} = \mu_0 \vec{j}_s. \tag{12}$$

Now, using the knowledge from before show that the time-independent Proca equation for a massive vector field,

$$\left[-\nabla^2 + \mu_0 \left(\frac{c}{\hbar}M\right)^2\right]\vec{A} = 0,\tag{13}$$

is fulfilled assuming an appropriate gauge (specify what you choose).

Remark: This procedure can also be done for the time-dependent case where the vector potential fulfils the inhomogeneous wave equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{A} = \vec{j}_s,\tag{14}$$

which corresponds to taking the surface current into account, and using the same reasoning as for the time-independent case, one can derive the Proca equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu_0 \left(\frac{c}{\hbar}M\right)^2\right]\vec{A} = 0,\tag{15}$$

which corresponds to the photons having mass.

e) *(2 points)* In the case of the Higgs mechanism, it is assumed that there is a vacuum field that suppresses the in principal long range force of the weak force exponentially, analogously to what happens in the superconductor with the in principle longe range force of the electromagnetic force. The W and Z boson have "effective" masses.

In natural units ($\hbar = c = \mu_0 = 1$), we can write the current $\vec{j}$ in the massive gauge boson case with the gauge field $\vec{A}$ as

$$\vec{j} = -g^2 v^2 \vec{A}\tag{16}$$

where g is the coupling strength and v is the minimum (vacuum) of the potential

$$V = \mu^2\phi^2 + \lambda\phi^4, \quad \lambda > 0\tag{17}$$

where μ and λ are real parameters of the Higgs potential and ϕ is the Higgs field. Which condition needs μ^2 to fulfil so that a non-vanishing v exists? What is the measured mass of the gauge boson? (You may use relations that have been useful before for relating the two pictures, the one of the mass and the one of the surface current.)

8 Black Hole Picture

In this problem, we will explore the recently unveiled image of the black hole M87*, recorded by the Event Horizon Telescope collaboration.

1. *(2 points)* Use Newtonian mechanics, together with the information that the speed of light is a universal "speed limit", to derive the radius of a black hole. Do not use Special or General Relativity here.
   Compare to the correct result from General Relativity, which says that in radial coordinates (so-called Schwarzschild coordinates), the radius of a black hole of mass M is $r_S = \frac{2GM}{c^2}$, where $G \approx 7 \cdot 10^{-11} Nm^2/kg^2$ is the Newton constant, and $c \approx 3 \cdot 10^8 m/s$ is the speed of light. Can you provide an argument why your result, even though it only uses Newtonian physics, gives roughly the correct answer? (hint: it might be useful to think about units to find an answer.)

2. *(3 points)* The black-hole has a "shadow" (dark region in the center of the image) because it "traps" light inside its event horizon, which is the Schwarzschild radius in the case of non-rotating black holes.
   Explain the existence of a light ring (the bright region around the shadow), based on the effect that massive compact objects act as "gravitational lenses", i.e., they bend light. In your explanation, take into account that the horizon (located at $r = r_s$ for a non-spinning black hole) is a *null hypersurface*, i.e., it is a surface in which light rays stay at constant r.

3. *(3 points)* To estimate the size of the telescope that is needed to resolve the shadow and light ring of M87*, first estimate the size of the shadow of M87*. Use 1) and the information that M 87* has a mass of about $6 \cdot 10^9 \, M_\odot$ ($M_\odot \approx 2 \cdot 10^{30} kg$) is a solar mass. Second, to estimate the required telescope size, use the distance to M87*, which is $d \approx 5 \cdot 10^{22} m$. Finally, take into account the information that the EHT observes at a wavelength of $\lambda \approx 1mm$, and that for a telescope of dish size L the resolution it can achieve is given by λ/L. Discuss your result in view of the fact that the EHT actually uses a collection of telescopes, located at the South Pole, in South America, North America and Europe.

4. *(2 points)* Observing the shadow of a black hole provides an unprecedented way of testing Einstein's theory of General Relativity (GR), because observing the shadow and comparing to predictions from GR tests gravity in the so-called "strong-field regime". Think about a
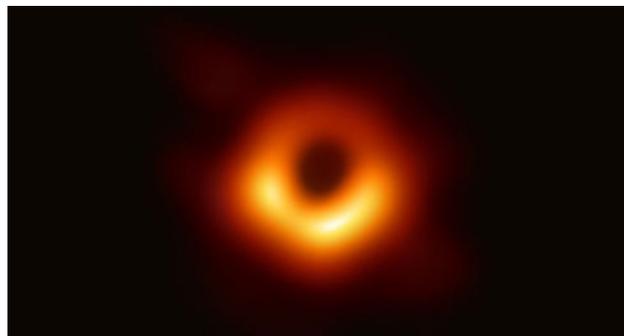

Figure 5: Image of M87*, Event Horizon Telescope Collaboration (The Astrophysical Journal Letters,875:L1(17pp), 2019 April 10).

way of quantifying this and compare to how "strong" gravity is in the solar system, where many observational tests of General Relativity already exist. (For a rough comparison, it might be useful to use the radius of the sun, $r_\odot \approx 7 \cdot 10^8 m$.)

9 Solid State Physics

The Fermi surface of a material is given by

$$\varepsilon(\mathbf{k}) = \hbar^2 \left(\frac{k_x^2 + k_y^2}{m^*} + \frac{k_z^2}{2m^*} \right)$$

where m^* and $2m^*$ are effective masses and $\mathbf{k} = (k_x, k_y, k_z)$ is the wavevector of the electron.

We consider an electron on the Fermi surface in the point $\mathbf{k}^0 = \left(k_x^0, k_y^0, k_z^0 \right)$

a) *(1 point) Determine the velocity vector for the electron* $\mathbf{v} = \frac{1}{\hbar} \nabla_{\mathbf{k}} \varepsilon(\mathbf{k})$ *(∇_k being the gradient with respect to* $\mathbf{v}$*)*

An electric field $\mathbf{E} = (E, 0, 0)$ is applied.

b) *(2 points) Determine the direction and magnitude for the acceleration of the electron*

Now the direction of the electric field is changed to $\mathbf{E} = \left(\frac{E}{\sqrt{2}}, 0, \frac{E}{\sqrt{2}} \right)$

c) *(2 points) Determine the new direction and magnitude for the acceleration of the electron*

Now the electric field $\mathbf{E} = \mathbf{0}$ and is replaced by a magnetic field $\mathbf{B} = (B, 0, 0)$

d) *(3 points) Show that*
$$\frac{dk_x}{dt} = 0 \ ; \ \frac{dk_y}{dt} = -\omega_1 dk_z; \ \ \frac{dk_z}{dt} = \omega_2 dk_y$$
and determine ω_1 *and* ω_2

e) *(2 points) Show from the result in d) that the electron oscillates in a plane perpendicular to* $\boldsymbol{B}$ *and determine the frequency of the oscillation.*

10 Freezing Front

In this problem we consider the freezing of water in a lake. When the temperature drops sufficiently the water begins to freeze and we wish to investigate the ice thickness with time.

The thickness of the ice is controlled by the rate at which the heat flow through the ice removes the latent heat released during freezing. As a first model, assume that the lake is infinite in extent and occupies the region $0 \leq x < \infty$ with the surface at $x = 0$. Also the water in the lake is at the freezing temperature, T_m, and the temperature distribution in the ice is linear. The temperature at the surface is constant, T_0. At time t the freezing front is at a distance, $X(t)$.

(a) *2 points* Obtain an expression for the rate of latent heat release of the liquid-solid transition.

(b) *2 points* Use the law of heat conduction to obtain an expression for the rate of flow of heat through the ice.

(c) *3 points* Hence show that the position of the freezing front is given by

$$X(t) = \sqrt{2Kt} \tag{18}$$

and find a formula for the constant K.

An analytic solution exists for this model

$$X(t) = 2\lambda\sqrt{\alpha t}$$

where $\alpha = \frac{k}{\rho c}$ and λ is the solution of the transcendental equation

$$\lambda e^{\lambda^2} \mathrm{erf}(\lambda) = \frac{c(T_m - T_0)}{L\sqrt{\pi}} \qquad \text{with} \qquad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \mathrm{du}$$

(d) *2 points* Use a first order approximation to find an expression for λ^2 and show that this also leads to the approximation (18).

(e) *1 points* This problem is set in terms of the freezing of water, how appropriate would it be as a first model for the solidification of molten steel?

Parameters
k Thermal conductivity, $Wm^{-1}K^{-1}$
L Specific latent heat of fusion, $J\mathrm{kg}^{-1}$
ρ Density, $\mathrm{kg}m^{-3}$
c Specific heat capacity, $J\mathrm{kg}^{-1}K^{-1}$

Values (in SI units	ice	steel
T_m	273	1650
ρ	900	7900
L	$3.3 \cdot 10^5$	$2.7 \cdot 10^5$
c	2100	450
k	2.2	40