Solidarity to achieve stability*

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Abstract

Agents may form coalitions. Each coalition shares its endowment among its agents by applying a sharing rule. The sharing rule induces a coalition formation problem by assuming that agents rank coalitions according to the allocation they obtain in the corresponding sharing problem. We characterize the sharing rules that induce a class of stable coalition formation problems as those that satisfy a natural axiom that formalizes the principle of solidarity. Thus, solidarity becomes a sufficient condition to achieve stability.

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Keywords: *solidarity, stability, coalition formation, sharing rule.*

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1 Introduction

According to the Merriam-Webster Dictionary, solidarity is defined as *unity (as of a group or class) that produces or is based on community of interests, objectives, and standards*. It is a fundamental ethical principle that can be traced back to ancient philosophers such as Socrates and Aristotle and it is also connected to the slogan of the French revolution. Nowadays, it even constitutes one of the six titles of the Charter of Fundamental Rights of the European Union.

The principle of solidarity has been often used in the axiomatic approach to economic design. It underlies many relational axioms formalizing a general idea: if the environment (e.g., resources, technology, population, or preferences) in which a group of people find themselves changes, and if no one in this group is responsible for the change, the welfare of all of them should be affected in the same direction: either they all end up at least as well off as they were initially, or they all end up at most as well off (Thomson, 2021). These axioms have been crucial to characterize egalitarian allocation rules in diverse settings (Roemer, 1986; Moulin, 1987; Moulin and Roemer, 1989; Moreno-Ternero and Roemer, 2006; Martínez and Moreno-Ternero, 2022). They have also been instrumental to characterize focal egalitarian rules in axiomatic bargaining and cooperative game theory (Kalai and Smorodinsky, 1975; Kalai, 1977; Young, 1988; Chun and Thomson, 1988). The egalitarian implications of the principle of solidarity are thus well explored. Here, we focus on less explored implications, highlighting its role as the mean to guarantee stability in contexts of coalition formation.

Coalition formation is the object of study of a large literature dealing with a plethora of social and economic problems such as cartel formation, lobbies, customs unions, conflict, public goods provision, political party formation, etc. (Ray, 2007; Grabisch and Funaki, 2012; Ray and Vohra, 2015). A central impetus in this literature goes to the notion of stability, which requires the immunity of a coalitional arrangement to "blocking" (Perry and Reny, 1994; Seidmann and Winter, 1998; Pulido and Sánchez-Soriano, 2006). In other words, a partition of agents into coalitions is (core) stable if there does not exist a counterfactual coalition whose members prefer to their current coalitions in the partition. We focus here on studying stability for environments in which coalition members have an endowment to be shared among them. In these contexts, we can define for each agent an individual preference over the possible coalitions she can be part of, depending on the sharing rule used to distribute the total endowment in each coalition: if the individual payoff an agent receives in one coalition is bigger than in another, then this individual will prefer the former coalition to the latter. As preferences are constructed on the basis of a sharing rule, we can refer to the coalition formation problem induced by the sharing rule. And the natural question that arises is whether we can find conditions on the sharing rule that guarantee the existence of stable partitions in the induced coalition formation problem. We actually show that such a condition is *solidarity*, formalized by the following axiom: the arrival of new agents to coalitions, whether or not this is accompanied by a change in the available endowment to share, should affect all the incumbent agents in the same direction.

Our solidarity axiom is equivalent to the combination of two axioms that appear frequently in the literature: *endowment monotonicity* and *consistency*. The former says that when a bad or good shock changes the endowment of a group, all its members should share in the calamity or windfall. Thus, it has obvious solidarity underpinnings and it has long been used in axiomatic work (Moulin and Thomson, 1988; Moulin, 1992; Moreno-Ternero and Vidal-Puga, 2021; Bergantiños and Moreno-Ternero, 2022). The latter says that if a sub-group of agents secedes with the endowment allocated to their members then, in the smaller economy, the rule allocates the remaining endowment in the same way. As such, it has normally been referred as a "robustness", "coherence", or "operational" principle (Balinski, 2005; Moreno-Ternero and Roemer, 2012; Thomson, 2019; Gudmundsson et al., 2023), although solidarity underpinnings have also been provided (Thomson, 2012). Alternative forms of consistency have indeed been suggested as axioms of stability in related contexts (Harsanyi, 1959; Lensberg, 1987, 1988).

Our main result actually shows that a sharing rule satisfies *solidarity* if and only if it induces *non-circular coalition formation problems*. Those are problems that preclude the existence of rings and satisfy the notion of *weak pairwise alignment*: if one agent in the intersection of two coalitions ranks them in one way, no other agent in the intersection ranks them in the opposite way.

The closest research to our work is Gallo and Inarra (2018), that combine coalition formation and claims problems (O'Neill, 1982). More precisely, they assume that agents have "claims" over the outputs they could produce by forming coalitions. Outputs cannot fully honor all claims and, thus, are rationed by a rule, which induces agents' preferences over coalitions. Within the domain of continuous rules, they show that the properties of endowment monotonicity and consistency guarantee the existence of stable partitions in the induced coalition formation problems.[1] We show that in a general context (without necessarily considering claims or imposing continuity) if the rule satisfies solidarity then stability is guaranteed for the induced coalition formation problem. Although the framework of both papers differs, our proofs rely on similar techniques. In particular, we will show that the solidarity property induces *non-circular* coalition formation problems, which they show to be enough for stability.

Another related paper is Pycia (2012). We shall be more precise about the connection once we formally introduce our result later in the text. But we mention at least now that Pycia (2012) analyzes a general model of coalition formation and shows therein that there is a stable coalition structure if agents' preferences are generated by a sharing rule consisting on Nash bargaining over coalitional outputs. More precisely, each rule within those satisfying two regularity conditions that guarantee stability can be represented by a profile of agents' functions. And, as in Nash (1950), the rule implies that members of each coalition share the output as if they were maximizing the product of their individual functions. In contrast, our result characterizes rules inducing stability without resorting to the regularity conditions mentioned above.

From a different viewpoint, Lensberg (1987) focuses on collective choice problems, which generalize the seminal model of axiomatic bargaining (Nash, 1950) to an arbitrary number of agents. A solution to these problems is a (collective) choice function that assigns to each element in a family of admissible collective choice problems a unique feasible utility allocation. Lensberg (1987) shows that a solution satisfying Pareto optimality, continuity, and consistency can be represented by an additively separable Bergson-Samuelson social welfare function.[2] That is, the functional form obtained by Lensberg (1987) coincides with the one obtained from a different perspective by Pycia (2012). Given that these sharing rules determined by this functional form *à la Nash* satisfy our property of solidarity, we can deduce as a byproduct

[1] Gallo and Inarra (2018) wrongly state that these properties are not only sufficient but also necessary.

[2] Lensberg (1988) shows that the Nash bargaining solution is actually characterized by Pareto optimality, anonymity, scale invariance, and consistency.

of our main result that this functional form guarantees solidarity in those settings.

The rest of the paper is organized as follows. In Section 2, we introduce the preliminaries of the model (sharing problems and coalition formation problems). In Section 3, we present our benchmark analysis and result. In Section 4, we present applications of our result to several focal problems such as bargaining, claims or ranking problems. Finally, we conclude in Section 5. For a smooth passage, the proofs have been relegated to the Appendix.

2 Preliminaries

2.1 Sharing problems

Let N be a finite set of agents. Consider a situation where a coalition of agents $C \subseteq N$ has an endowment $E \in \mathbb{R}_+$. A **sharing problem** is a pair $(C, E) \in N \times \mathbb{R}_+$. Let $\boldsymbol{\mathcal{P}}$ denote the class of such problems.

An **allocation** for $(C, E) \in \mathcal{P}$ is a vector $x = (x_i)_{i \in C} \in \mathbb{R}_+^{|C|}$ that satisfies non-negativity, $0 \leq x_i$ for each $i \in C$, and efficiency, $\sum_{i \in C} x_i = E$. A **sharing rule** is a function F defined on $\mathcal{P}$ that associates with each $(C, E) \in \mathcal{P}$ an allocation $F(C, E)$ for (C, E). The payoff of agent i in problem (C, E) under rule F is denoted by $F_i(C, E)$. We denote by $\boldsymbol{\mathcal{F}}$ the set of all sharing rules.

We now introduce several axioms for sharing rules.

The first axiom states that small changes in the endowment of the problem do not lead to large changes in the chosen allocation.

Endowment continuity: For each sharing problem $(C, E) \in \mathcal{P}$ and each sequence of endowments $\{E_j\}_{j=1}^{\infty}$ with $E_j \to E$,

$$F(C, E_j) \to F(C, E).$$

The second axiom states that if the endowment increases, then each agent receives at least as much as she initially did.

Endowment monotonicity: For each pair of sharing problems (C, E), $(C, E') \in \mathcal{P}$, with $E < E'$, and each $i \in C$,

$$F_i(C, E) \leq F_i(C, E').$$

The third axiom states that if some agents leave the coalition with their payoffs (provided by the rule), and the situation is reassessed, then each remaining agent in the coalition receives the same payoff as she initially did.

Consistency: For each sharing problem $(C, E) \in \mathcal{P}$, each $C' \subset C$, and each $i \in C'$,

$$F_i(C', \sum_{i \in C'} F_i(C, E)) = F_i(C, E).$$

We finally introduce the axiom of solidarity. This axiom states that the arrival of new agents to a coalition (with or without changes in the endowment) does not affect the incumbent agents in different directions.[3]

Solidarity: For each pair of sharing problems $(C, E), (C', E') \in \mathcal{P}$, with $C \subset C'$, and each pair $i, j \in C$,

$$F_i(C, E) > F_i(C', E') \Rightarrow F_j(C, E) \geq F_j(C', E').$$

The next lemma, whose standard proof we omit, states some relations between the previous axioms.[4]

Lemma 1. *The following statements hold:*

- *If a sharing rule satisfies endowment monotonicity, then it also satisfies endowment continuity.*
- *A sharing rule satisfies solidarity if and only if it satisfies endowment monotonicity and consistency.*

2.2 Coalition formation problems

Consider a situation where each agent ranks the coalitions that she may belong to. Formally, let N be a finite set of agents and $C \subseteq N$ denote a coalition. The collection of non-empty coalitions is denoted by 2^N. For each agent $i \in N$, let $\succsim_i$ be a complete and transitive preference relation over coalitions containing i. Given $C, C' \subseteq N$ such that $i \in C \cap C'$, $C \succsim_i C'$ means that agent i finds coalition C at least as desirable as coalition C'. The binary

[3]This axiom is related to the property of population-and-resource monotonicity introduced by Chun (1999) for claims problems. See also Moreno-Ternero and Roemer (2006).

[4]In particular, the proof of the second statement follows a similar reasoning than in Chun (1999).

relations $\succ_i$ and $\sim_i$ are defined as usual. A **coalition formation problem (or a preference profile)** consists of a list of preference relations, one for each $i \in N$, $\succsim = (\succsim_i)_{i \in N}$. Let $\boldsymbol{\mathcal{D}}$ denote the class of such problems.

A partition is a set of non-empty coalitions whose union is N and whose pairwise intersections are empty. Formally, a **partition** is a list $\pi = \{C_1, \ldots, C_m\}$ such that *(i)* for each $l = 1, \ldots, m$, $C_l \neq \emptyset$, *(ii)* $\bigcup_{l=1}^{m} C_l = N$, and *(iii)* for each pair $l, l' \in \{1, \ldots, m\}$, with $l \neq l'$, $C_l \cap C_{l'} = \emptyset$. Let $\mathbf{\Pi}(N)$ denote the set of all partitions. For each $\pi \in \Pi(N)$ and each $i \in N$, let $\boldsymbol{\pi(i)}$ denote the coalition in π which contains agent i. A partition $\pi \in \Pi$ is **stable** for $\succsim$ if there is no coalition $T \subseteq N$ such that for each $i \in T$, $T \succ_i \pi(i)$. The set of all stable partitions for $\succsim$ is the **core** of $\succsim$. The literature on coalition formation mostly focuses on identifying properties on the preference profiles that guarantee the existence of stable partitions.[5]

We now introduce several concepts and properties defined for preference profiles. We first introduce the concept of a ring.[6] A ***ring*** is an ordered list of coalitions $(C_1, \ldots, C_l)$, with $l > 2$, such that for each $k = 1, \ldots, l$ (subscript modulo l) and each $j \in C_k \cap C_{k+1}$, $C_{k+1} \succsim_j C_k$, with at least one agent with strict preference in each intersection. That is, in a ring there is at least one agent in the intersection of any two consecutive coalitions with a strict preference of the later coalition over the former, while the rest of the intersection-mates can be indifferent, but no one can prefer the former.

This next property, originally introduced by Pycia (2012), requires that all agents in the intersection of two coalitions rank them in the same way.

Pairwise alignment: A preference profile $\succsim \in \mathcal{D}$ is pairwise aligned if for each pair $C, C' \subseteq N$ and each pair $i, j \in C \cap C'$, then $[C \succsim_i C' \Leftrightarrow C \succsim_j C']$.

Farrell and Scotchmer (1988) introduce the ***common ranking property*** for preferences profiles. This property states that there is a common ranking of all coalitions that agrees with agents' preferences.[7] Indeed, when all coalitions are feasible, the common ranking property coincides with pairwise alignment (see Footnote 6 in Pycia (2012) for more details).

[5]See, for instance, Banerjee et al. (2001) and Bogomolnaia and Jackson (2002) for different sufficient conditions that guarantee stability.

[6]See, for instance, Inal (2015) and Pycia (2012) for different definitions of rings, under the name of cycles.

[7]Formally, a coalition formation problem satisfies the common ranking property if there is an ordering $\succsim$ over $2^N \setminus \{\emptyset\}$ such that for each $i \in N$ and each $C, C' \subseteq N$ with $i \in C \cap C'$, $C \succsim_i C' \Leftrightarrow C \succsim C'$.

A weakening of the *pairwise alignment* property, introduced by Gallo and Inarra (2018), requires that if one agent in the intersection of two coalitions ranks them in one way, no other agent in the intersection ranks them in the opposite way.

Weak pairwise alignment: A preference profile $\succsim \in \mathcal{D}$ is weakly pairwise aligned if for each pair $C, C' \subseteq N$ and each pair $i, j \in C \cap C'$, then $[C \succ_i C' \Rightarrow C \succsim_j C']$.

Note that, unlike pairwise alignment, weak pairwise alignment allows one agent to have a strict preference over two coalitions while any other agent in the intersection is indifferent between them.

Gallo and Inarra (2018) introduce the class of coalition formation problems that satisfy weak pairwise alignment and do not have rings, and call them ***non-circular coalition formation problems***. This class includes the problems that satisfy the common ranking property (the proof is straightforward) and is contained in the class of problems that satisfy the ***top coalition property*** (see Theorem 1 in Gallo and Inarra, 2018).[8] Note that these properties are sufficient conditions for stability. The relations among all the above-mentioned properties are illustrated in Figure 1.

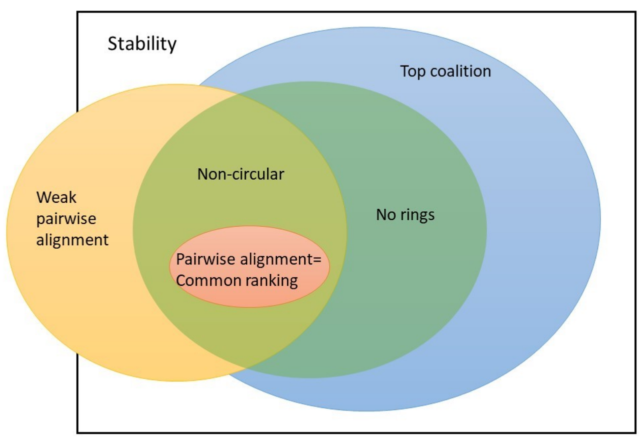

Figure 1: Relations among properties.

[8]Formally, let $C \subseteq N$. A coalition $C' \subseteq C$ is a top coalition of C if for each $i \in C'$ and each $S \subseteq C$ with $i \in S$, we have $C' \succsim_i S$. A coalition formation problem satisfies the top coalition property if each coalition $C \subseteq N$ has a top coalition (Banerjee et al., 2001).

3 The benchmark analysis

Given a set of sharing problems, one for each coalition, and a sharing rule, a coalition formation problem can be induced as follows: each agent computes her payoff in each sharing problem with the sharing rule, and ranks coalitions accordingly. Formally, given a set of sharing problems $\{(C, E_C)\}_{C \subseteq N}$, the **coalition formation problem induced by $\boldsymbol{F} \in \mathcal{F}$** is the list of preference relations $\succsim^F = (\succsim_i^F)_{i \in N}$ defined as follows: for each $i \in N$, and each pair $C, C' \subseteq N$ such that $i \in C \cap C'$, $C \succsim_i^F C'$ if and only if $F_i(C, E_C) \geq F_i(C', E_{C'})$.

Our main result characterizes all rules that induce non-circular coalition formation problems. They happen to be those that satisfy the solidarity axiom. The proof can be found in the Appendix.

Theorem 1. *A sharing rule satisfies solidarity if and only if it induces a non-circular coalition formation problem.*

The next result follows from Theorem 1 and the relations among properties presented above (see Figure 1).

Corollary 1. *If a sharing rule F satisfies solidarity, then for any set of sharing problems $\{(C, E_C)\}_{C \subseteq N}$, the core of the coalition formation problem $\succsim^F$ is non-empty.*

Corollary 1 implies that, when the sharing rule satisfies solidarity, then stability is guaranteed. These results are illustrated in Figure 2.

Theorem 1 is related to the results in Pycia (2012). This author analyzes a model of coalition formation including environments in which not all coalitions are feasible so that many-to-one matching problems with externalities are also considered. He provides a preference domain in which pairwise alignment is a necessary and sufficient condition for stability (Theorems 1 and 2). As an application, he also shows that sharing rules obeying strict endowment monotonicity[9] and non-satiability[10] generate preference profiles in that domain, i.e., preference profiles satisfying pairwise alignment (Corollary 1).[11]

[9]Formally, for each pair of sharing problems $(C, E), (C, E') \in \mathcal{P}$, with $E < E'$, and each $i \in N$, $F_i(C, E) < F_i(C, E')$.

[10]Formally, $\lim_{E \to \infty} F_i(C, E) = \infty$ for all $C \subseteq N$ and all $i \in C$.

[11]He also includes the axiom of *endowment continuity*, but this axiom is implied by *strict endowment monotonicity*, thanks to our Lemma 1 presented above.

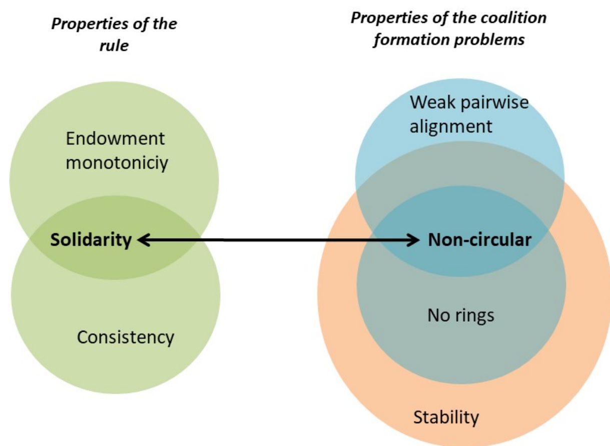

Figure 2: Sharing rules and their induced coalition formation problems.

By contrast, we restrict our study to one-sided coalition formation problems. We consider the preference domain of the non-circular problems in which pairwise alignment is weakened and absence of rings is imposed. Then, we show that the only sharing rules that generate preference profiles in that domain satisfy solidarity. Consequently, if the sharing rule satisfies this axiom the existence of stable partitions for the induced coalition formation problems is guaranteed. These are not only minor technical differences because, as we shall illustrate in the following example, some interesting sharing rules inducing stability are covered by our result, but not by Pycia's one.

Example 1. *Consider a sharing rule reflecting a situation in which an agent has priority over the rest of the agents. Moreover, she has a satiation level of k units of the endowment, while the rest of the society never satiates. Then, F allocates the first k units of the endowment of any coalition to this prioritized agent (if she is in the coalition), while the remaining units of the endowment (if left) are shared equally among the other agents within the coalition. If the prioritized agent is not in the coalition, the rule simply imposes equal sharing of the endowment among all coalition members.*

Formally, let $N = \{1, \ldots, n\}$ and the following sharing rule:

$$F_i(C,E) = \begin{cases} \min\{E,k\} & \textit{if } i = 1 \in C, \\ \frac{E-\min\{E,k\}}{|C|-1} & \textit{if } 1, i \in C \textit{ and } i \neq 1, \\ \frac{E}{|C|} & \textit{if } 1 \notin C \textit{ and } i \in C. \end{cases}$$

It can be checked that this sharing rule satisfies solidarity and then, by our Theorem 1, the rule also induces stability. We illustrate this result by showing the existence of a stable partition for a particular specification. Let $N = \{1, 2, 3, 4\}$*,* $k = 10$*, and the distribution of endowments be defined as follows:*

C	$\{12\}$	$\{13\}$	$\{14\}$	$\{123\}$	$\{124\}$	$\{134\}$	$\{1234\}$	otherwise
E_C	10	12	14	16	20	24	28	0

Then, F *yields the following allocations:*

C	$\{12\}$	$\{13\}$	$\{14\}$	$\{123\}$	$\{124\}$	$\{134\}$	$\{1234\}$
$F(C, E_C)$	$(10,0)$	$(10,2)$	$(10,4)$	$(10,3,3)$	$(10,5,5)$	$(10,7,7)$	$(10,6,6,6)$

As a consequence, the coalition formation problem induced by F*,* $\succsim^F$*, is the following:*

1	2	3	4
$12 \sim 13 \sim 14 \sim$	1234	134	134
$\sim 123 \sim 124 \sim$	124	1234	1234
$\sim 134 \sim 1234$	123	123	124
1	12	13	14
	$2 \sim 23 \sim$	$3 \sim 23 \sim$	$4 \sim 24 \sim$
	$\sim 24 \sim 234$	$\sim 34 \sim 234$	$\sim 34 \sim 234$

Note that partitions $\{\{134\}, \{2\}\}$ *and* $\{\{1234\}\}$ *are stable. This example shows that* $\succsim^F$ *does not necessarily satisfy pairwise alignment; observe that, for instance,* $\{124\} \succ_2^F \{123\}$*, whereas* $\{124\} \sim_1^F \{123\}$*. However, it satisfies weak pairwise alignment and it has no rings, i.e., it is a non-circular problem. The reason why this sharing rule is not included in Pycia's results is that it does satisfy neither strict endowment monotonicity nor non-satiability.*[12]

[12]It can be checked that the coalition formation problems that F can generate (with all possible endowment vectors) do not satisfy the "richness" conditions (see Section 3.2 in Pycia, 2012) that are necessary for the results in that paper.

The next example shows that, although the *solidarity* axiom is sufficient to guarantee stability in the induced coalition formation problems, it is not necessary.

Example 2. *Consider a variant of Example 1, reflecting a situation in which agent* 1 *has priority over the rest of the agents, but only when the grand coalition is formed. Moreover, as before, she has a satiation level of* k *units of the endowment, while the rest of the society never satiates. Then,* F *allocates the first* k *units of the endowment of the grand coalition to agent* 1*, while the remaining units of the endowment (if left) are shared equally among the other agents. If the coalition is not the grand coalition, the rule simply imposes equal sharing of the endowment among all coalition members.*

Formally, let $N = \{1, \ldots, n\}$ *and the following sharing rule:*

$$F_i(C, E) = \begin{cases} \min\{E, k\} & \text{if } i = 1 \text{ and } C = N, \\ \frac{E - \min\{E,k\}}{|C|-1} & \text{if } i \neq 1, \text{and } C = N, \\ \frac{E}{|C|} & \text{otherwise }. \end{cases}$$

It can be checked that this sharing rule does not satisfy solidarity; in particular, it is not consistent. However, it never generates rings and, therefore, it induces stability. To see this, note that the sharing rule imposes equal sharing for each coalition $C \neq N$*. Thus, the common ranking property is satisfied by all coalitions but coalition* N*. Then, adding coalition* N *to the preference profile cannot generate a ring (as this would require at least three coalitions).*

We illustrate this rule for a particular specification. Let $N = \{1, 2, 3\}$*,* $k = 6$*, and the distribution of endowments be defined as follows:*

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$	otherwise
E_C	10	8	6	15	0

Then, F *yields the following allocation in each coalition:*

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$
$F(C, E_C)$	$(5, 5)$	$(4, 4)$	$(3, 3)$	$(6, 4.5, 4.5)$

As a consequence, the coalition formation problem induced by F*,* $\succsim^F$*, is the following:*

1	2	3
123	12	123
12	123	13
13	23	23
1	2	3

Note that partitions $\{\{12\},\{3\}\}$ and $\{\{123\}\}$ are stable. However, $\succsim^F$ does not satisfy weakly pairwise alignment as $\{123\} \succ_1^F \{12\}$, whereas $\{12\} \succ_2^F \{123\}$.

4 Applications

In many economic models, agents are characterized by different features (such as utility functions, claims, or ranking positions) that could be taken into account to distribute a given endowment among them. Our results show that, regardless of these characteristics, as long as the sharing rule satisfies solidarity, it will induce a coalition formation problem with a non-empty core. In this section we first develop an application based on Dietzenbacher and Kondratev (2022) and then we relate our result to Gallo and Inarra (2018) and Pycia (2012).

4.1 Ranking problems

Dietzenbacher and Kondratev (2022) introduce the problem of prize allocation in competitions. In this model, there is a prize endowment to be shared among the participants of a competition according to their ranking. With this idea in mind, we propose a model where agents are ranked and all coalitions can be formed. Then, the rule may take the ranking of the agents into account to derive the final individual payoffs. This model can be applied to any setting where agents can be ordered according to some characteristic (such as their expertise or past performance).

Formally, let $N = \{1, ..., n\}$ be the set of agents. A **ranking** $\mathcal{R}$ is a bijection $\mathcal{R} : N \longrightarrow \{1, ..., n\}$ that assigns to each agent a position, i.e., $\mathcal{R}(i)$ is the position of agent i in the ranking. We say that agent $i \in N$ has a higher position in the ranking than agent $j \in N$ if $\mathcal{R}(i) < \mathcal{R}(j)$. For each $C \subseteq N$, let E_C denote the endowment of coalition C and $\mathcal{R}_C$ the projection of ranking $\mathcal{R}$

to C. That is, the position of agent i in coalition C is the number of agents, including herself, that have a higher position in that coalition. Formally, for each $i \in C$, $\mathcal{R}_C(i) = |\{j \in C : \mathcal{R}(i) \leq \mathcal{R}(j)\}|$. For each coalition $C \subseteq N$, denote by $(C, E_C, \mathcal{R}_C)$ the **ranking problem** for coalition C. An allocation for $(C, E_C, \mathcal{R}_C)$ is a vector $x = (x_i)_{i \in C} \in \mathbb{R}_+^{|C|}$ such that, for each $i \in C$, $0 \leq x_i$, and $\sum_{i \in C} x_i = E_C$. A sharing rule G, defined as in Section 2.1., is a function that associates with each $(C, E_C, \mathcal{R}_C)$ an allocation. Given a set of ranking problems $\{(C, E_C, \mathcal{R}_C)\}_{C \subseteq N}$, the **coalition formation problem induced by rule $\boldsymbol{G}$**, $\succsim^G$, is defined as in Section 3.

Based on the family of *interval rules* considered in Dietzenbacher and Kondratev (2022), we reformulate this family for each coalitional ranking problem as follows:

Interval rules for ranking problems: There exist disjoint intervals $(a_1, b_1), (a_2, b_2), \ldots$ with $a_1, a_2, \dots \in \mathbb{R}_+$ and $b_1, b_2, \dots \in \mathbb{R}_+ \cup \{+\infty\}$ such that for each problem $(C, E_C, \mathcal{R}_C)$,

$$G_i(C, E_C, \mathcal{R}_C) = \begin{cases} a_k & if \quad na_k \leq E_C \leq (n-\beta)a_k + \beta b_k; \\ x & if \quad (n-\beta)a_k + \beta b_k \leq E_C \leq (n - \mathcal{R}_C(i))a_k + \mathcal{R}_C(i) b_k; \\ b_k & if \quad (n - \mathcal{R}_C(i))a_k + \mathcal{R}_C(i) b_k \leq E_C \leq nb_k; \\ \frac{E_C}{n} & if \quad \text{otherwise,} \end{cases}$$

where $n = |N|$, $\beta = \mathcal{R}_C(i) - 1$ and $x = E - (n - \mathcal{R}_C(i))a_k - \beta b_k$.

As Dietzenbacher and Kondratev (2022) mention, the interval rule with $a_k = b_k = 0$ for each k coincides with the Equal Division while the interval rule with $a_1 = 0$ and $b_1 = +\infty$ coincides with the Winner Takes All, both, two well-known rules.

Theorem 1 in Dietzenbacher and Kondratev (2022) states that these are the only order-preserving[13] rules that satisfy solidarity.[14] Consequently, our Theorem 1 yields the following.

Corollary 2. *The interval rules are the only order-preserving rules for ranking problems that induce non-circular coalition formation problems.*

Corollary 2 also implies that the interval rules guarantee stability in the induced coalition formation problem. Other interesting rules proposed by

[13] If $\mathcal{R}(i) < \mathcal{R}(j)$, then $G_i(C, E_C, \mathcal{R}_C) \geq G_j(C, E_C, \mathcal{R}_C)$.

[14] Dietzenbacher and Kondratev (2022) do not use the axiom of *solidarity*, but the separate axioms of *endowment monotonicity* and *consistency*.

Dietzenbacher and Kondratev (2022) do not yield stability. The following example illustrates these features.

Example 3. *Let $N = \{1, 2, 3\}$ be the set of agents such that $\mathcal{R}(i) = i$ for each $i \in N$. Assume the following coalitional endowments:*

C	$\{12\}$	$\{23\}$	$\{13\}$	$\{123\}$
E_C	20	14	15	21

Consider first an interval rule G with $(a_1, b_1) = (2, 9)$, $(a_2, b_2) = (9, 10.5)$ and $(a_3, b_3) = (10.5, +\infty)$. We obtain the following individual payoffs:

C	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$
$G(C, E_C, \mathcal{R}_C)$	$(10.5, 9.5)$	$(9, 6)$	$(9, 5)$	$(9, 9, 3)$

The induced coalition formation problem, $\succsim^G$, is the following:

1	2	3
12	12	13
$13 \sim 123$	$23 \sim 123$	23
1	2	123
		3

Observe that the partition $\{\{12\}, \{3\}\}$ is stable.

We finally consider a class of rules for ranking problems based on the family of proportional rules defined in Dietzenbacher and Kondratev (2022). Formally, let $\lambda^1, \lambda^2, \ldots \lambda^{|N|} \in \mathbb{R}_+$ be such that $\lambda^1 > 0$ and $\lambda^k \geq \lambda^{k+1}$, for each $k \in \{1, \ldots, |N| - 1\}$. For each problem $(C, E_C, \mathcal{R}_C)$,

$$H_i(C, E_C, \mathcal{R}_C) = \frac{\lambda^{\mathcal{R}_C(i)}}{\sum_{j \in C} \lambda^{\mathcal{R}_C(j)}} \cdot E_C.$$

There exist rules within this family that do not satisfy solidarity and do not guarantee stability. An instance is the rule obtained when $\lambda^1 = 3$ and $\lambda^2 = \lambda^3 = 1$, for which we have the following individual payoffs:

C	$\{12\}$	$\{23\}$	$\{13\}$	$\{123\}$
$H(C, E_C, \mathcal{R}_C)$	$(15, 5)$	$(10.5, 3.5)$	$(11.25, 3.75)$	$(12.6, 4.2, 4.2)$

The induced coalition formation problem, $\succsim^H$, is the following:

1	2	3
12	23	123
123	12	13
13	123	23
1	2	3

Note that this problem is not a non-circular coalition formation problem. In particular, coalitions $(\{12\}, \{23\}, \{13\})$ *form a ring and as* $23 \succ_2 123$*, while* $123 \succ_3 23$*, weak pairwise alignment is also violated. Moreover, it can easily be shown that it has an empty core.*

4.2 Other related problems

We now relate our results to other settings previously analyzed in the literature.

For instance, Theorem 1 can be applied to the case of coalition formation in claims problems. This model, introduced by Gallo and Inarra (2018), considers a situation where agents have claims over the endowment and rules take those claims into account to get the allocations.[15] They show that, among others, each of the so-called *parametric rules* for claims problems (see Young, 1987; Stovall, 2014) guarantees stability, whereas others such as the so-called *random arrival rule* (see O'Neill, 1982; Thomson, 2019) do not. To be more precise, they actually prove that the parametric rules, which satisfy solidarity, induce non-circular coalition formation problems. This can also be obtained from our Theorem 1. They also show that the random arrival rule, which fails to satisfy consistency, can generate a coalition formation problem with an empty core for certain coalitional endowments (see Section 2.4 in Gallo and Inarra, 2018).

We also consider the case of coalition formation in bargaining problems introduced by Pycia (2012). In this model, agents have utility functions which

[15]This model is renamed as generalized claims problems by Gallo and Klaus (2022).

may be taken into account by the rule to get the final allocations. He shows that the so-called Nash bargaining solution (Nash, 1950) guarantees stability, whereas the so-called Kalai-Smorodinsky bargaining solution (Kalai and Smorodinsky, 1975) does not. The fact that the Nash bargaining solution satisfies solidarity guarantees (as stated in our Theorem 1) that it induces non-circular coalition formation problems and, thus, stability. By contrast, the Kalai-Smorodinsky bargaining solution fails to satisfy consistency giving rise to coalition formation problems with an empty core for some coalitional endowments (see Section 2 in Pycia, 2012).

5 Final remarks

We have studied in this paper coalition formation problems in a context in which coalitions have to share collective resources. We have characterized the sharing rules that induce a non-circular coalition formation problem as those satisfying a natural axiom formalizing the principle of solidarity. This implies that such a solidarity axiom guarantees core-stable partitions in the induced coalition formation problem. Our result can be applied to canonical problems of resource allocation long studied such as bargaining, or claims problems as well as to other problems recently considered such as ranking problems.

There exist other problems of resource allocation in which our result cannot be directly applied. Instances are network games (e.g., Jackson and Wolinsky, 2003; Jackson, 2005), river sharing problems (e.g., Ambec and Sprumont, 2002; Alcalde-Unzu et al., 2015), or revenue sharing in hierarchies (e.g., Hougaard et al., 2017; Harless, 2020). In these problems not all coalitions are feasible and, consequently, the definition of consistency, as considered in this paper, could not be directly applied. However, we believe that extending to these cases the connection we highlight here between solidarity (in the resource allocation problem) and stability (in the corresponding coalition formation problem) is worth exploring in further research.

Appendix: Proof of Theorem 1

We first present an auxiliary lemma that shows that if the rule applied to a sharing problem satisfies solidarity, then it can allocate the same payoffs to its agents in an extended sharing problem with new added agents.

Lemma 2. *Let F be a sharing rule that satisfies solidarity. Then, for each pair $C, C' \subseteq N$, with $C' \subset C$, and each $E' \in \mathbb{R}_+$, there exists $E \in \mathbb{R}_+$ such that for each $i \in C'$, $F_i(C, E) = F_i(C', E')$.*

Proof. Let F be a sharing rule that satisfies *solidarity*. Then, by Lemma 1, F satisfies *endowment continuity* and *consistency*. Let $C, C' \subseteq N$, with $C' \subset C$. We construct $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $\xi \in \mathbb{R}_+$, $\alpha(\xi) = \sum_{i \in C'} F_i(C, \xi)$. As F satisfies *endowment continuity*, α is continuous. Then, for each $E' \in \mathbb{R}_+$, there exists $E \in \mathbb{R}_+$ such that $\alpha(E) = E'$. By *consistency*, $F_i(C, E) = F_i(C', E')$ for each $i \in C'$, as desired. □

We now show that if F satisfies solidarity, then $\succsim^F$ always satisfies weak pairwise alignment.

Lemma 3. *If F satisfies solidarity, then $\succsim^F$ satisfies weak pairwise alignment for each $\{E_C\}_{C \subseteq N}$.*

Proof. Let F be a sharing rule that satisfies *solidarity*. Then, by Lemma 1, F satisfies *endowment monotonicity* and *consistency*. Let $C, C' \subseteq N$, $E_C, E_{C'} \in \mathbb{R}_+$, and $i, j \in C \cap C'$.

If $C \subset C'$ or $C' \subset C$, then by *solidarity*, either $[F_k(C, E_C) \leq F_k(C', E_{C'})$ for each $k \in \{i, j\}]$ or $[F_k(C, E_C) \geq F_k(C', E_{C'})$ for each $k \in \{i, j\}]$. Therefore, agents i and j do not rank C and C' in opposite ways.

Otherwise, $C \not\subset C'$ and $C' \not\subset C$. Then, let $x_k = F_k(C, E_C)$ for each $k \in C$, and $x'_{k'} = F_{k'}(C', E_{C'})$ for each $k' \in C'$. Consider the sharing problems $(\{i, j\}, x_i + x_j), (\{i, j\}, x'_i + x'_j)$. By *consistency*,

$$(x_i, x_j) = F(\{i, j\}, x_i + x_j) \text{ and } (x'_i, x'_j) = F(\{i, j\}, x'_i + x'_j).$$

Assume, without loss of generality, that $x_i + x_j \geq x'_i + x'_j$. Then, by *endowment monotonicity*, for each $k \in \{i, j\}$, $x_k = F_k(\{i, j\}, x_i + x_j) \geq F_k(\{i, j\}, x'_i + x'_j) = x'_k$. Therefore, for each $k \in \{i, j\}$, $F_k(C, E_C) \geq F_k(C', E_{C'})$. Consequently, agents i and j do not rank C and C' in opposite ways.

Hence, $\succsim^F$ satisfies weak pairwise alignment, as desired. □

We now show that if $\succsim^F$ satisfies weak pairwise alignment, then the grand coalition can never be part of any ring of $\succsim^F$.

Lemma 4. *If $\succsim^F$ satisfies weak pairwise alignment for each $\{E_C\}_{C\subseteq N}$, then N is not part of any ring of $\succsim^F$.*

Proof. Let $\succsim^F$ be a coalition formation problem that satisfies weak pairwise alignment. Suppose by contradiction that there exists $\{E_C\}_{C\subseteq N}$ such that $\succsim^F$ has a ring $(C_1,\ldots,C_l)$ with $N = C_k$ for some $k \in \{1,\ldots,l\}$. Then, for each $k = 1,\ldots,l$ (subscript modulo l), there is at least one agent, say agent $j_{k+1} \in C_{k+1} \cap C_k$, such that $C_{k+1} \succ^F_{j_{k+1}} C_k$. Note that, by transitivity of the agents' preferences, $j_1 = \ldots = j_l$ is not possible.

Assume without loss of generality that $N = C_1$. Then, $N \succ^F_{j_1} C_l$ and $C_2 \succ^F_{j_2} N$. Observe that, by the structure of the ring, $C_3 \succ^F_{j_3} C_2$. Furthermore, since $\succsim^F$ satisfies weak pairwise alignment, $C_2 \succsim^F_{j_3} N$. Then, by transitivity, $C_3 \succ^F_{j_3} N$. In a similar way, for each $k > 3$, we have that $C_k \succ^F_{j_k} N$. If $j_l = j_1$, we have that $N \succ^F_{j_1} C_l$ and $C_l \succ^F_{j_1} N$, which contradicts transitivity. If $j_l \neq j_1$, we have that $N \succ^F_{j_1} C_l$ and $C_l \succ^F_{j_l} N$, which contradicts that $\succsim^F$ satisfies weak pairwise alignment. $\square$

Next, we prove that if F satisfies solidarity, then it is guaranteed that $\succsim^F$ generates no rings.

Lemma 5. *If F satisfies solidarity, then for each $\{E_C\}_{C\subseteq N}$, $\succsim^F$ has no rings.*

Proof. Let F be a sharing rule that satisfies *solidarity*. Suppose by contradiction that there exists $\{E_C\}_{C\subseteq N}$ such that $\succsim^F$ has a ring $(C_1,\ldots,C_l)$. Then, for each $k = 1,\ldots,l$ (subscript modulo l), there is at least one agent, say agent $j_{k+1} \in C_{k+1} \cap C_k$, such that $C_{k+1} \succ^F_{j_{k+1}} C_k$. Given Lemma 4, $N \neq C_k$ for each $k \in \{1,\ldots,l\}$.

Consider now $\{E'_C\}_{C\subseteq N}$ such that $E'_C = E_C$ for each $C \subset N$ and E'_N is such that $F_i(N, E'_N) = F_i(C_1, E_{C_1})$ for each $i \in C_1$ (whose existence is guaranteed by Lemma 2). We denote by $\succsim^{F'}$ the coalition formation problem when F is applied and the endowments are $\{E'_C\}_{C\subseteq N}$.

By construction, for each $i \in C_1$, $F_i(N, E'_N) = F_i(C_1, E'_{C_1})$ and, therefore, $C_1 \sim^{F'}_i N$. In particular, $C_1 \sim^{F'}_{j_1} N$ and $C_1 \sim^{F'}_{j_2} N$ (possibly $j_1 = j_2$). Similarly, for each $i' \in C_k \cap C_{k+1}$, $k = 1,\ldots,l$ (subscript modulo l), $F_{i'}(C_k, E'_{C_k}) = F_{i'}(C_k, E_{C_k})$ and $F_{i'}(C_{k+1}, E'_{C_{k+1}}) = F_{i'}(C_{k+1}, E_{C_{k+1}})$. In particular, for each $k = 1,\ldots,l$ (subscript modulo l), $C_{k+1} \succ^{F'}_{j_{k+1}} C_k$. Then, by transitivity,

$N \succ_{j_1}^{F'} C_l$ and $C_2 \succ_{j_2}^{F'} N$. As F satisfies *solidarity*, $\succsim^{F'}$ satisfies weak pairwise alignment by Lemma 3 and, therefore, $C_2 \succsim_{j_3}^{F'} N$. Given that $C_3 \succ_{j_3}^{F'} C_2$, we have that, by transitivity, $C_3 \succ_{j_3}^{F'} N$. In a similar way, for each $k > 3$, we have that $C_k \succ_{j_k}^{F'} N$. If $j_l = j_1$, $N \succ_{j_1}^{F'} C_l$ and $C_l \succ_{j_1}^{F'} N$, which contradicts transitivity. If $j_l \neq j_1$, $N \succ_{j_1}^{F'} C_l$ and $C_l \succ_{j_l}^{F'} N$, which contradicts that $\succsim^{F'}$ satisfies weak pairwise alignment and then, by Lemma 3, F does not satisfy *solidarity*. $\square$

Lemmas 3 and 5 prove one implication of Theorem 1, while the other is proven by the following lemma.

Lemma 6. *If F does not satisfy solidarity, then there is $\{E_C\}_{C \subseteq N}$ such that $\succsim^F$ does not satisfy weak pairwise alignment.*

Proof. Let F be a sharing rule that does not satisfy *solidarity*. Then, there exist $C, C' \subseteq N$, with $C' \subset C$, $i, j \in C'$ and $E_C, E_{C'} \in \mathbb{R}_+$ such that $F_i(C, E_C) > F_i(C', E_{C'})$ and $F_j(C, E_C) < F_j(C', E_{C'})$. Then, for $\{E_C\}_{C \subseteq N}$, we have that $C \succ_i^F C'$ and $C' \succ_j^F C$. Hence, $\succsim^F$ does not satisfy weak pairwise alignment. $\square$

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