Notes on

Chapter 3: Oligopoly

Microeconomic Theory IV

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Introduction

Non-Cooperative Game Theory is very useful for modelling and understanding the *multi-agent* economic problems characterized by *strategic interdependency*, in particular for analyzing competition between firms in a market. Perfect competition and pure monopoly (not threatened by entry) are special non-realistic cases. It is more frequent in the real life to find industries with not many firms or with a lot of firms but with a smaller number of them producing a large proportion of the total production. With few firms, competition is characterized by strategic considerations: each firm takes its decisions (price, output, advertising, etc.) taking into account or conjecturing the behaviour of the others. Competition in an oligopoly can be seen as a game where firms are the players. So we shall adopt a Game Theory perspective to analyze the different models of oligopoly. For each case, we shall wonder what game firms are playing (information, order of playing, strategies, etc.), and what the equilibrium notion is. An important difference between the games of the previous chapter and the games we shall solve in this chapter is that the former were finite games while the latter are infinite games.

1.1. **The Cournot model**

1.1.1. ***Duopoly***

(i) Context.

(ii) Representation of the game in normal form.

(iii) Notion of equilibrium.

(iv) Best response function. Characterization of equilibrium.

(v) Example. Graphic representation.

(i) *Context*

The Cournot model has four basic characteristics:

a) We consider a market served by *2 firms*.

b) *Homogeneous product*. That is, from the consumers' point of view, the goods produced by the two firms are perfect substitutes.

c) *Quantity competition.* The variable of choice of each firm is output level. Denote by x_1 and x_2 the production levels of firm 1 and firm 2, respectively.

d) *Simultaneous choice*. The two firms have to choose their outputs simultaneously. That is, each firm has to choose its output without knowing the rival's choice. Simultaneous Choice does not mean that choices are made at the same instant in time. An equivalent context would be one where one firm chooses its output first and tthe other firm chooses its output afterwards but without observing the decision adopted by the first firm. In other words, sequential choice together with imperfect information (the player moving second does not observe what the first mover does) is equivalent to simultaneous choice.

The inverse demand function is $p(x)$, where $p^{'}(x) < 0$. As the product is homogeneous, the price at which a firm can sell will depend on the total output: $p(x) = p(x_1 + x_2)$.

The production cost of firm i is $C_i(x_i)$, i=1,2.

(ii) *Representation of the game in normal form*

1) $i=1,2.$ (Players)

2) $x_i \geq 0.$ Any non-negative amount would serve as a strategy for player *i*. Equivalently, we can represent the set of strategies of player *I* as $x_i \in [0,\infty),\ i=1,2.$

3) The payoff obtained by each firm at the combination of strategies (x_1,x_2) is:

$$\left.\begin{array}{l}\Pi_1(x_1,x_2)=p(x_1+x_2)x_1-C_1(x_1)\\ \Pi_2(x_1,x_2)=p(x_1+x_2)x_2-C_2(x_2)\end{array}\right\}\equiv \Pi_i(x_i,x_j)=p(x_i+x_j)x_i-C_i(x_i),\ i,j=1,2,\ j\neq i.$$

(iii) *Notion of equilibrium. Cournot-Nash equilibrium*

It is very easy to adapt the definition of the Nash equilibrium in the previous chapter to this new context.

“ $s^*\equiv(s_1^*,..,s_n^*)$ is a Nash equilibrium if : $\Pi_i(s_i^*,s_{-i}^*)\geq\Pi_i(s_i,s_{-i}^*)\ \forall s_i\in S_i,\forall i,i=1,...,n$.”

For the Cournot duopoly game we say:

“ (x_1^*,x_2^*) is a Cournot-Nash equilibrium if $\Pi_i(x_i^*,x_j^*)\geq\Pi_i(x_i,x_j^*)\ \forall x_i\geq 0, i,\ j=1,2,\ j\neq i$ ”.

The second definition based on best responses proves more useful.

“ $s^*\equiv(s_1^*,..,s_n^*)$ is a Nash equilibrium if: $s_i^*\in MR_i(s_{-i}^*)\ \forall i, i=1,..,n$ where $MR_i(s_{-i}^*)=\left\{s_i^{'}\in S_i:\Pi_i(s_i^{'},s_{-i}^*)\geq\Pi_i(s_i,s_{-i}^*),\ \forall s_i\in S_i, s_i\neq s_i^{'}\right\}$”.

For the Cournot duopoly game:

"(x_1^*, x_2^*) is a Cournot-Nash equilibrium if $x_i^* = f_i(x_j^*),\ i, j = 1, 2,\ j \neq i$",

where $f_i(x_j)$ is the best-response function of firm *i* to firm *j*'s output.

(iv) *Best response function. Characterization of equilibrium*

The proceedure that we follow to obtain the Nash equilibrium is similar to that used in the previous chapter. First we calculate the best response of each player to the possible strategies of its rival and then we look for combinations of strategies which are mutually best responses to each other.

Given a strategy of firm *j* we look for the strategy that provides most profit for firm *i*. That is,, given the strategy $x_j \geq 0$ firm *i*'s best response is to choose a strategy x_i such that:

$$\max_{x_i \geq 0} \Pi_i(x_i, x_j) \equiv p(x_i + x_j)x_i - C_i(x_i)$$

$$\frac{\partial \Pi_i}{\partial x_i} = p(x_i + x_j) + x_i p'(x_i + x_j) - C_i'(x_i) = 0 \ \ (1) \to \overline{f}_i(x_j)$$

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = 2p'(x_i + x_j) + x_i p''(x_i + x_j) - C_i''(x_i) < 0$$

Taking into account the non-negativity constraint, $x_i \geq 0,$ or in terms of game theory taking into account that the best response must belong to the strategy space of the player, the best response function is: $f_i(x_j) = \max\left\{\overline{f}_i(x_j), 0\right\}$.

The Cournot-Nash equilibrium is a strategy profile (x_1^*, x_2^*) such that the strategy of each player is its best response to the rival's strategy. That is,

$$\left.\begin{array}{l} x_1^* = f_1(x_2^*) = \max\left\{\overline{f}_1(x_2^*), 0\right\} \\ \\ x_2^* = f_2(x_1^*) = \max\left\{\overline{f}_2(x_1^*), 0\right\} \end{array}\right\} \leftrightarrow x_i^* = f_i(x_j^*) = \max\left\{\overline{f}_i(x_j^*), 0\right\},\ i, j = 1, 2,\ j \neq i.$$

Let us now forget the non-negativity constraint and we are going to assume that the best response function is broadly characterized by condition (1) (interior solution). By definition, the best response must satisfy the first order condition: $\frac{\partial \Pi_i(f_i(x_j),x_j)}{\partial x_i}=0 \rightarrow$ firm *i*'s best response to $x_j \geq 0$ is $f_i(x_j)$. The Cournot-Nash equilibrium satisfies $\frac{\partial \Pi_i(x_i^*,x_j^*)}{\partial x_i}=0$ given that $x_i^* = f_i(x_j^*),\ i=1,2.$ We have a simple way of checking whether a combination of strategies is a Nash equilibrium: by calculating each firm's marginal profit corresponding to that strategy profile. If any one of them is other than zero the equilibrium condition is not met.

$\frac{\partial \Pi_i(\hat{x}_i,\hat{x}_j)}{\partial x_i} > 0 \rightarrow f_i(\hat{x}_j) > \hat{x}_i \rightarrow (\hat{x}_i,\hat{x}_j)$ is not a Cournot-Nash equilibrium.

$\frac{\partial \Pi_i(\hat{x}_i,\hat{x}_j)}{\partial x_i} < 0 \rightarrow f_i(\hat{x}_j) < \hat{x}_i \rightarrow (\hat{x}_i,\hat{x}_j)$ is not a Cournot-Nash equilibrium.

(v) *Example. Graphic representation*

Consider the case of linear demand and constant marginal cost: $p(x)=a-bx$ and $C_i(x_i)=c_i x_i,\ i=1,2.$ Assume for the sake of simplicity that the marginal cost is the same for the two firms: $c_i = c > 0,\ i=1,2.$ ($a > c$ for the example to make sense).

We first obtain the best response function for firm $i,\ i=1,2.$

$$\max_{x_i \geq 0} \Pi_i(x_i,x_j) \equiv p(x_i+x_j)x_i - C_i(x_i) \equiv [a-b(x_i+x_j)]x_i - cx_i \equiv [a-c-b(x_i+x_j)]x_i$$

$$\frac{\partial \Pi_i}{\partial x_i} = p(x_i+x_j)+x_i p'(x_i+x_j) - C_i'(x_i) = a-2bx_i-bx_j-c=0 \rightarrow \overline{f}_i(x_j)\frac{a-c-bx_j}{2b}$$

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = -2b < 0$$

So the best response function is:

$$f_i(x_j)=\max\left\{\overline{f}_i(x_j),0\right\}=\max\left\{\frac{a-c-bx_j}{2b},0\right\}.$$

The Cournot-Nash equilibrium satisfies:

$$x_1^* = f_1(x_2^*) = \max\left\{\frac{a-c-bx_2^*}{2b}, 0\right\} \underbrace{\geq 0}_{\text{given that } a>c}$$

$$x_2^* = f_2(x_1^*) = \max\left\{\frac{a-c-bx_1^*}{2b}, 0\right\} \underbrace{\geq 0}_{\text{given that } a>c}$$

By solving the system:

$$x_1^* = f_1(x_2^*) = f_1(\underbrace{f_2(x_1^*)}_{x_2^*})$$

$$x_1^* = \frac{a-c-bx_2^*}{2b} = \frac{a-c-b\left(\frac{a-c-bx_1^*}{2b}\right)}{2b} = \frac{\frac{a-c+bx_1^*}{2}}{2b} = \frac{a-c+bx_1^*}{4b} \to x_1^* = \frac{a-c}{3b}.$$

$$\to x_2^* = \frac{a-c-bx_1^*}{2b} = \frac{a-c-b\left(\frac{a-c}{3b}\right)}{2b} = \frac{2(a-c)}{6b} = \frac{a-c}{3b}.$$

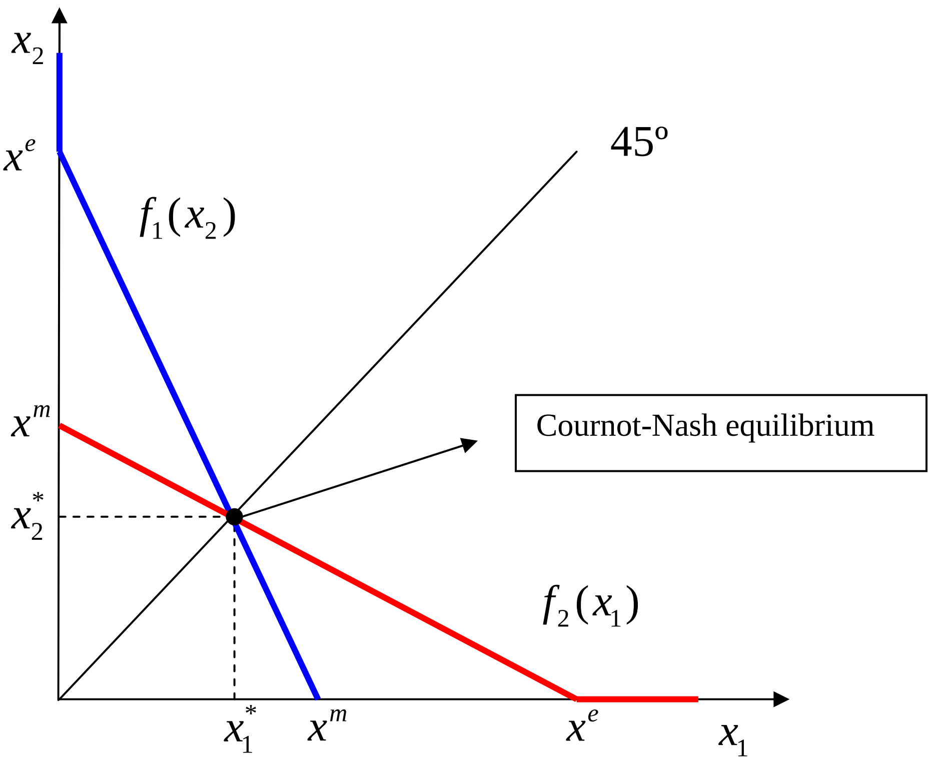
The Cournot-Nash total output is: $x^* = x_1^* + x_2^* = \frac{2(a-c)}{3b}$ and the equilibrium price

$p^* = p(x_1^* + x_2^*) = a - b\frac{2(a-c)}{3b} = \frac{a+2c}{3}$. Profits are:

$$\Pi_1^* = \Pi_1(x_1^*, x_2^*) = [p(x_1^* + x_2^*) - c]x_1^* = \frac{a-c}{3}\frac{a-c}{3b} = \frac{(a-c)^2}{9b}$$

$$\Pi_2^* = \Pi_2(x_1^*, x_2^*) = [p(x_1^* + x_2^*) - c]x_2^* = \frac{a-c}{3}\frac{a-c}{3b} = \frac{(a-c)^2}{9b}.$$

Graphic representation



1.1.2. ***Oligopoly***

(i) Representation of the game in normal form.

(ii) Notion of equilibrium. Best response function. Cournot-Nash equilibrium.

(iii) Lerner index.

(iv) Special cases. Constant marginal cost.

(i) *Representation of the game in normal form*

1) $i = 1, 2, ..., n.$ (Players)

2) $x_i \geq 0.$ Similarly, $x_i \in [0, \infty),\ i = 1, 2, .., n.$

3) The profit of each firm corresponding to strategy profile (x_i, x_{-i}) is:

$$\Pi_i(x_i, x_{-i}) = p(\underbrace{x_i + x_{-i}}_{x})x_i - C_i(x_i),\ i = 1, 2, ..., n.$$

The way of representing the game in normal form has changed slightly. Given the strategy profile $(x_1, x_2, ..., x_n)$ the relevant point for firm i, $i = 1, 2, ..., n,$ is the total output produced by the other firms, $x_{-i} = \sum_{j \neq i} x_j$. Therefore, (x_i, x_{-i}) is not really a strategy profile and $\Pi_i(x_i, x_{-i})$ is firm *i*'s profit associated with every combination of strategies where firm *i* produces x_i and the other firms produce in aggregate x_{-i} (it being irrelevant to firm *i* how that production x_{-i} is distributed among the *n* -1 firms).

(iii) *Notion of equilibrium. Best response functions. Cournot-Nash equilibrium*

In the Cournot oligopoly game we say that

"$(x_1^*, x_2^*, .., x_n^*) \equiv (x_i^*, x_{-i}^*)$ is a Cournot-Nash equilibrium if:

$$\Pi_i(x_i^*, x_{-i}^*) \geq \Pi_i(x_i, x_{-i}^*)\ \forall x_i \geq 0,\ \forall i, i = 1, 2, ..., n."$$

In terms of best response functions the definition is:

"$(x_1^*, x_2^*, .., x_n^*) \equiv (x_i^*, x_{-i}^*)$ is a Cournot-Nash equilibrium if $x_i^* = f_i(x_{-i}^*), \ \forall i, i = 1, 2, .., n.$",

where $f_i(x_{-i})$ is firm *i*'s best response function to all those combinations of strategies whose total output is x_{-i}.

We next obtain the best response of firm *i* to all those combinations of strategies (of the other firms) whose total output is x_{-i}. Firm *I*'s best response is to choose a strategy x_i such that:

$$\max_{x_i \geq 0} \Pi_i(x_i, x_{-i}) \equiv p(x_i + x_{-i})x_i - C_i(x_i)$$

$$\frac{\partial \Pi_i}{\partial x_i} = p(x_i + x_{-i}) + x_i p'(x_i + x_{-i}) - C_i'(x_i) = 0 \quad (1) \rightarrow \overline{f}_i(x_{-i})$$

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = 2p'(x_i + x_{-i}) + x_i p''(x_i + x_{-i}) - C_i''(x_i) < 0$$

Taking into account the non-negativity constraint, $x_i \geq 0$, or in terms of game theory that the best response must belong to the player's strategy space, the best response function is: $f_i(x_{-i}) = \max\left\{\overline{f}_i(x_{-i}), 0\right\}$.

The Cournot-Nash equilibrium is a strategy profile $(x_1^*, x_2^*, .., x_n^*) \equiv (x_i^*, x_{-i}^*)$ such that $x_i^* = f_i(x_{-i}^*), \ \forall i, i = 1, 2, .., n.$".

Let us now forget the non-negativity constraint and assume that the best response function is broadly characterized by condition (1) (interior solution). By definition, the best response must satisfy the first order condition: $\dfrac{\partial \Pi_i(f_i(x_{-i}), x_{-i})}{\partial x_i} = 0 \rightarrow$ firm *i*'s best response to

$x_{-i} \geq 0$ is $f_i(x_{-i})$. The Cournot-Nash equilibrium satisfies $\frac{\partial \Pi_i(x_i^*, x_{-i}^*)}{\partial x_i} = 0$ given that $x_i^* = f_i(x_{-i}^*)$, $i = 1, 2, ..., n$. Here again there is a simple way of checking whether a combination of strategies is a Nash equilibrium: calculating each firm's marginal profit corresponding to that strategy profile. If any one is other than zero the equilibrium condition is not met.

$\frac{\partial \Pi_i(\hat{x}_i, \hat{x}_{-i})}{\partial x_i} > 0 \rightarrow f_i(\hat{x}_{-i}) > \hat{x}_i \rightarrow (\hat{x}_i, \hat{x}_{-i})$ is not a Cournot-Nash equilibrium.

$\frac{\partial \Pi_i(\hat{x}_i, \hat{x}_{-i})}{\partial x_i} < 0 \rightarrow f_i(\hat{x}_{-i}) < \hat{x}_i \rightarrow (\hat{x}_i, \hat{x}_{-i})$ is not a Cournot-Nash equilibrium.

(iii) *Lerner index*

By assuming an interior solution we are going to transform condition (1) to obtain the Lerner index of market power.

$$p(\underbrace{x_i + x_{-i}}_{x}) + x_i p'(x_i + x_{-i}) - C_i'(x_i) = 0$$

$$p(x)[1 + x_i \frac{p'(x)}{p(x)}] - C_i'(x_i) = 0$$

$$p(x)[1 + \frac{x_i}{x} \underbrace{\frac{xp'(x)}{p(x)}}_{-\frac{1}{|\varepsilon(x)|}}] - C_i'(x_i) = 0$$

Defining firm *i*'s share as $s_i = \frac{x_i}{x}$ we get:

$$p(x)[1 - \frac{s_i}{|\varepsilon(x)|}] - C_i'(x_i) = 0$$

Then the Lerner index of market power for firm *i* is

$$\frac{p(x) - C_i'(x_i)}{p(x)} = \frac{s_i}{|\varepsilon(x)|}$$

Then the Cournot model is located between the case of pure monopoly ($s_i = 1$) and perfect competition ($\lim_{s_i \to 0} \frac{p - C^{'}}{p} = 0$).

(iv) *Special cases*. *Constant marginal costs*

a) **Constant marginal cost:** $c_i > 0,\ i = 1,..,n.$

At equilibrium the first order condition for each firm (interior solution) must be satisfied:

$$p(\underbrace{x_i^* + x_{-i}^*}_{x^*}) + x_i^* p^{'}(x_i^* + x_{-i}^*) - c_i = 0 \quad i = 1,2..,n.$$

By adding up the n first order conditions:

$$np(x^*) + \underbrace{\sum_{i=1}^{n} x_i^*}_{x^*} p^{'}(x^*) - \sum_{i=1}^{n} c_i = 0$$

That is

$$np(x^*) + x^* p^{'}(x^*) = \sum_{i=1}^{n} c_i$$

Then the total output in Cournot-Nash equilibrium depends exclusively on the sum of the marginal costs, not on their distribution across firms (in an interior solution with all n firms producing positive quantities).

b) **Common constant marginal cost:** $c_i = c > 0,\ i = 1,..,n.$

The Lerner index is:

$$\frac{p(x) - c}{p(x)} = \frac{s_i}{|\varepsilon(x)|}$$

Taking into account that if the product is homogeneous and the marginal cost is the same for all firms then the Cournot-Nash equilibrium should be symmetric:

$$s_i = \frac{x_i^*}{x^*} = \frac{\overline{x}^*}{n\overline{x}^*} = \frac{1}{n}, \quad i = 1,..,n.$$

If the price-elasticity is constant then:

$$\frac{p(x) - c}{p(x)} = \frac{1}{n|\varepsilon|}$$

Therefore, when the number of firms increases the relative price-marginal cost margin (the Lerner index) decreases and at the limit when $n \to \infty$, $p \to c$.

1.1.3. ***Welfare analysis***

We consider the simplest case where marginal cost is constant and common to all firms.

$$p(\underbrace{x_i^* + x_{-i}^*}_{x^*}) + x_i^* p'(x_i^* + x_{-i}^*) - c = 0 \quad i = 1, 2.., n.$$

By adding the *n* first order conditions:

$$np(x^*) + x^* p'(x^*) - nc = 0$$

We follow a similar approach to that in the chapter on monopolies to compare the Cournot output with the efficient output.

(*Review the obtaining of the welfare function and the problem of maximizing social welfare*)

$$\max_{x \geq 0} W(x) \equiv \max_{x \geq 0} u(x) - C(x)$$

$$W'(0) = u'(0) - C'(0) > 0 \Rightarrow \; p(0) > C'(0)$$

$$W'(x) = u'(x) - C'(x) = 0 \Leftrightarrow \; W'(x^e) = 0 \quad \text{First order condition.}$$

$$W''(x) = u''(x) - C''(x) < 0 \; \text{Strictly concave welfare function.}$$

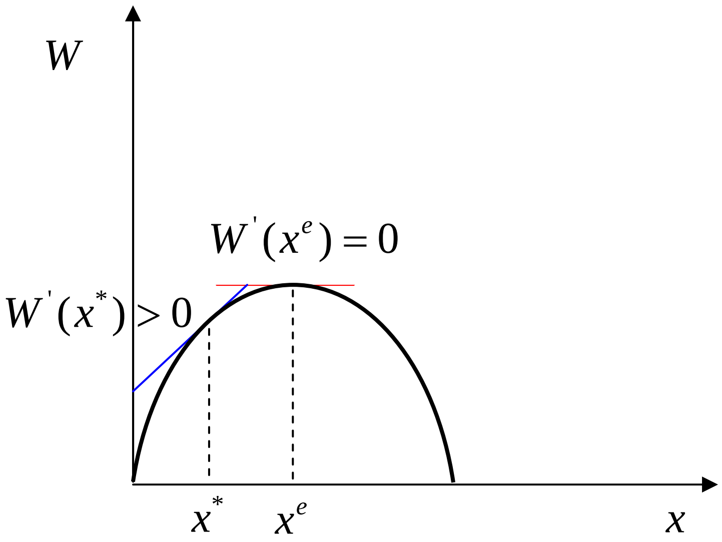
$$\left\{ \begin{matrix} W'(x^e) = 0 \\ W'(x^*)? \\ W''(x) < 0 \end{matrix} \right\}$$

$$W'(x^*) = \underbrace{u'(x^*)}_{p(x^*)} - C'(x^*) = -\frac{x^*}{n} \underbrace{\overbrace{p'(x^*)}^{u''(x^*)}}_{<0} > 0$$

By definition of Cournot output.

$$\left\{ \begin{matrix} W'(x^e) = 0 \\ W'(x^*) > 0 \\ W''(x) < 0 \end{matrix} \right\} \rightarrow W'(x^e) < W'(x^*) \rightarrow \; x^e > x^*$$

$$W''(x) < 0 \Leftrightarrow \frac{dW'(x)}{dx} < 0 \rightarrow \uparrow x \; \downarrow W'(x)$$



W
$W^{'}(x^{e})=0$
$W^{'}(x^{*})>0$
x^{*}
x^{e}
x

1.2. **The Bertrand model**

1.2.1. ***Homogeneous product***

(i) Context.

(ii) Residual demand.

(iii) Representation of the game in normal form. Notion of equilibrium.

(iv) The Bertrand Paradox. Characterization of equilibrium and uniqueness.

(i) *Context*

The Bertrand model is characterized by the following elements

1) We consider a market with *2 firms*.

2) Firms sell a *homogeneous product*.

3) *Price competition*.

4) *Simultaneous choice*. Each firm has to choose its price with no knowledge of the rival's decision. Again, simultaneous choice does not mean that choices are made at the same instant in time; the relevant point is that although one firm may play first the other does not observe its decision.

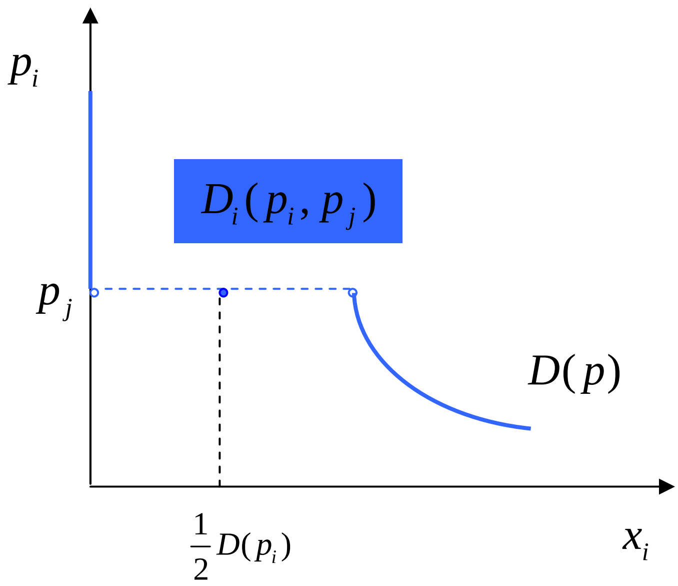
5) *Constant marginal cost and common*: $c_1 = c_2 = c > 0$.

(ii) *Residual demand*

Firms produce a homogeneous product and compete on price. Then from the consumers' point of view the only relevant point is the relationship between the prices of the two firms; consumers buy the product from the firm that sets the lower price. That is, if one firm charges a lower price than the other, it captures the entire market and the second firm sells nothing. If both firms give the same price then consumers are indifferent between buying from one or the other. For the sake of simplicity we assume that in the case of equal prices each firm sells to the half of the market.

Firm *i*'s residual demand, $i, j = 1, 2,\ j \neq i,$ is:

$$D_i(p_i, p_j) = \begin{cases} D(p_i) & p_i < p_j \\ \frac{1}{2}D(p_i) & p_i = p_j \\ 0 & p_i > p_j \end{cases}$$



(iii) *Representation of the game in normal form. Notion of equilibrium*

The game in **normal form** is:

1) $i = 1, 2.$ (Players)

2) $p_i \geq 0.$ Any non negative price serves as a strategy for player *i*. Equivalently, we can represent the player *i*'s strategy space as $p_i \in [0, \infty),\ i = 1, 2.$

3) The firms profits corresponding to the strategy profile (p_1, p_2) are:

$$\left.\begin{array}{l} \Pi_1(p_1, p_2) = (p_1 - c)D_1(p_1, p_2) \\ \Pi_2(p_1, p_2) = (p_2 - c)D_2(p_1, p_2) \end{array}\right\} \equiv \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j),\ i, j = 1, 2,\ j \neq i,$$

where the residual demand for firm *i*, $i, j = 1, 2,\ j \neq i,$ is:

$$D_i(p_i, p_j) = \begin{cases} D(p_i) & p_i < p_j \\ \frac{1}{2}D(p_i) & p_i = p_j \,. \\ 0 & p_i > p_j \end{cases}$$

In the Bertrand game we say that

"(p_1^*, p_2^*) is a Bertrand-Nash equilibrium if:

$$\Pi_i(p_i^*, p_j^*) \geq \Pi_i(p_i, p_j^*) \ \forall p_i \geq 0,\ i, j = 1, 2,\ j \neq i \text{ ”.}$$

To simplify the analysis we use this definition exclusively because the fact that the residual demand of each firm is a discontinuos function of its own price means that we cannot use standard optimization techniques (in fact instead of having best response functions we would have best response correspondences and the analysis would be more complex).

(iv) *The Bertrand Paradox. Characterization of the equilibrium and uniqueness*

We demonstrate here that the unique Bertrand-Nash equilibrium is:

$$p_1^* = p_2^* = c$$

This result is known as the *Bertrand Paradox*:

"*Two firms competing on prices suffice to obtain a competitive outcome*"

Demonstration

We demonstrate that the strategy profile $p_1^* = p_2^* = c$:

a) is a Nash equilibrium.

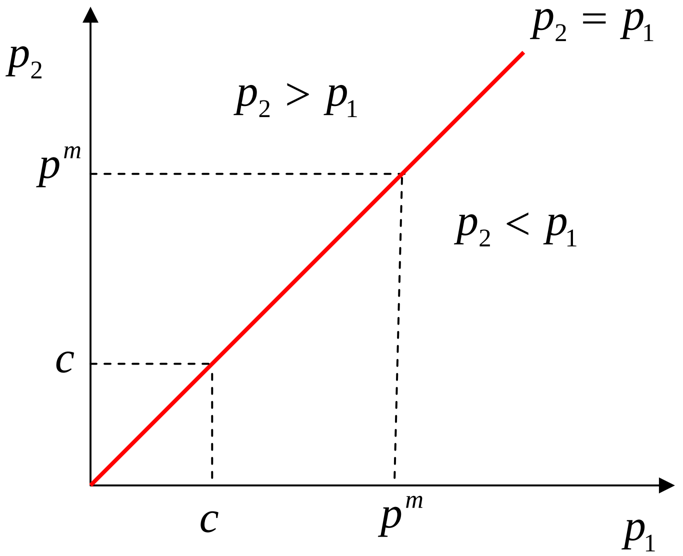
b) is the unique Nash equilibrium.

a) The profit of each firm under strategy profile (c, c) is: $\Pi_i(c,c) = (c-c)\frac{1}{2}D(c),\ i = 1, 2.$

If firm *i* unilaterally deviates by charging a price $p_i > c$ then its profit will be zero because it will make no sales. By reducing its price below marginal cost $p_i < c$ it would capture the entire market but incur losses. Therefore,

$$\Pi_i(c,c) \geq \Pi_i(p_i, c) \ \forall p_i \geq 0,\ i, j = 1, 2,\ j \neq i$$

b) We demonstrate that no other combination of strategies can be a Nash equilibrium.The graph below shows the different types of strategy profile. We check whether a strategy profile is an equilibrium or not by calculating the profit of each player corresponding to this combination of strategies and we wonder if any of the players has an incentive to unilaterally deviate. To eliminate a strategy profile as an equilibrium it suffices to show that at least one player can improve by deviating unilaterally.



1) Equal prices: $p_i = p_j$

a) ¿ $p_i = p_j > c$ NE? NO. In a strategy profile like this the profit of each firm would be: $\Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = (p_i - c)\frac{1}{2}D(p_i)$. Firm i =1, 2 would have an incentive to deviate unilaterally. For example, we can choose a price $p_i^{'} = p_i - \varepsilon$ (where ε is an arbitrary positive amount as small as required):

$$(p_i^{'} - c)D(p_i^{'}) = (p_i^{'} - c)D_i(p_i^{'}, p_j) = \Pi_i(p_i^{'}, p_j) > \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = (p_i - c)\frac{1}{2}D(p_i).$$

In fact there would be inifinite deviations such that firm i improves with a unilateral deviation.

b) ¿ $p_i = p_j < c$ NE? NO. Firm *i*'s profit in such a strategy profile would be: $\Pi_i(p_i, p_j) = \underbrace{(p_i - c)}_{<0} D_i(p_i, p_j) = (p_i - c)\frac{1}{2}D(p_i) < 0.$ Firm *i* would have an incentive to deviate unilaterally. For example, any price $p_i^{'} > p_i$:

$$0 = (p_i^{'} - c)\underbrace{D_i(p_i^{'}, p_j)}_{=0} = \Pi_i(p_i^{'}, p_j) > \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = (p_i - c)\frac{1}{2}D(p_i).$$

2) Different prices: $p_i \neq p_j$

c) ¿ $p_i > p_j > c$ NE? NO. Firm *i*'s profit in such strategy profile would be zero: $\Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = 0$ and the profit of the other firm, firm *j*, would be $\Pi_j(p_i, p_j) = (p_j - c)D_j(p_i, p_j) = (p_j - c)D(p_j) > 0.$ For firm *i* any unilateral deviation $p_i^{'}$ such that $c < p_i^{'} \leq p_j$ increases profits:

$$(p_i^{'} - c)\underbrace{D(p_i^{'})}_{\text{si } p_i^{'} < p_j} = (p_i^{'} - c)D_i(p_i^{'}, p_j) = \Pi_i(p_i^{'}, p_j) > \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j) = (p_i - c)0 = 0.$$

Although we have already proved that the strategy profile (p_i, p_j), with $p_i > p_j > c$ cannot be an equilibrium we can also show that in many cases firm *j* would also have an incentive to deviate unilaterally. (For example, if $p^m \geq p_i > p_j > c$ any unilateral deviation $p_i > p_j^{'} > p_j$ increases the profits of firm *j*. For the cases $p_i > p_j > p^m > c$ and $p_i > p^m > p_j > c$ it is also straightforward to find deviations such the profits of firm *j* increase. The only situation where firm *j* would have no incentive to deviate would be one such that $p_i > p^m = p_j > c$).

d) Other cases:

- ¿ $p_i > c \geq p_j$ NE? NO. Firm *i* would have no incentive to deviate unilaterally while for firm *j* when $p_i > c > p_j$ any $p_j^{'} > p_j$ increases profits and when $p_i > c = p_j$ firm *j* increases profits by conveniently increasing the price. For example, if $p^m \geq p_i > c = p_j$ any price $p_i > p_j^{'} > c$ increases the profits of firm *j*. When $p_i > p^m > c = p_j$ any price $p^m > p_j^{'} > c$ (and others) increases the profits of firm *j*.

- ¿ $c \geq p_i > p_j$ NE? NO. Firm *i* would have no incentive to deviate while for firm *j* any price $p_j^{'} > p_j$ increases profits.

1.2.2. ***Heterogeneous products*** (differentiated products)

(i) Heterogeneous product. Residual demand.

(ii) Representation of the game in normal form.

(iii) Notion of equilibrium. Best response function. Bertrand-Nash equilibrium.

(i) *Heterogeneous product.Residual demand*

We maintain the rest of the assumptions of the Bertrand model (two firms, simultaneous choice, constant and common marginal cost, price competition) but now we consider that the two firms sell heterogeneous products. That is, firms sell products that are close but imperfect substitutes.

The demand for the product of firm *i*, the residual demand, is given by $D_i(p_i, p_j)$. Assume that $\frac{\partial D_i}{\partial p_i} < 0$, $\frac{\partial D_i}{\partial p_j} > 0$ and $\left|\frac{\partial D_i}{\partial p_i}\right| > \frac{\partial D_i}{\partial p_j}$; that is, the demand for product *i* is a decreasing function of its own price, products are substitutes and the own effect is larger than the cross effect.

(iii)*Representation of the game in normal form. Notion of equilibrium*

The game in normal form is:

1) $i = 1, 2.$ (Players)

2) $p_i \geq 0$ or equivalently $p_i \in [0, \infty),\ i = 1, 2.$

3) The profit of each firm corresponding to (p_1, p_2) is:

$$\left.\begin{array}{l} \Pi_1(p_1, p_2) = (p_1 - c)D_1(p_1, p_2) \\ \\ \Pi_2(p_1, p_2) = (p_2 - c)D_2(p_1, p_2) \end{array}\right\} \equiv \Pi_i(p_i, p_j) = (p_i - c)D_i(p_i, p_j),\ i, j = 1, 2,\ j \neq i$$

Now the residual demand of each firm is a continous function of its price.

(iv) *Notion of equilibrium*. *Best response function. Bertrand-Nash equilibrium*

In terms of best responses the definition of the Bertrand-Nash equilibrium is:

"(p_1^*, p_2^*) is a Bertrand-Nash equilibrium if $p_i^* = g_i(p_j^*),\ \forall i, j = 1, 2,\ j \neq i.$",

where $g_i(p_j)$ is firm *i*'s best response to the price p_j of its rival.

The best response of firm *i* consists of choosing p_i such that:

$$\max_{p_i \geq 0} \Pi_i(p_i, p_j) \equiv (p_i - c)D_i(p_i, p_j)$$

$$\frac{\partial \Pi_i}{\partial p_i} = D_i(p_i, p_j) + (p_i - c)\frac{\partial D_i}{\partial p_i} = 0 \quad (1) \rightarrow g_i(p_j)$$

$$\frac{\partial^2 \Pi_i}{\partial p_i^2} = 2\frac{\partial D_i}{\partial p_i} + (p_i - c)\frac{\partial^2 D_i}{\partial p_i^2} < 0.$$

1.3. **Leadership in the choice of output. The Stackelberg model**

(i) Context.

(ii) Two-stage game. Perfect information. Notion of strategy.

(iii) Backward induction. Subgame perfect equilibrium.

(iv) Example: linear demand and constant marginal cost.

(v) Other Nash equilibria which are not subgame perfect.

(i) *Context*

The Stackelberg duopoly model has four basic characteristics:

a) We consider a market with *2 firms*.

b) *Homogeneous product*. That is, from the consumers' point of view the two firms produce products which are perfect substitutes.

c) *Quantity competition*. Let x_1 and x_2 be the outputs of firm 1 and 2, respectively.

d) *Sequential choice*. One of the firms (the leader), firm 1, chooses its output level first. Next the other firm (the follower), firm 2, chooses its output after observing the output chosen by firm 1. From a game theory perspective this is a perfect information game.

(ii) *A two-stage game. Perfect information. Notion of strategy*

Firms play a two-stage game:

Stage 1: firm 1 chooses its output $x_1 \geq 0$.

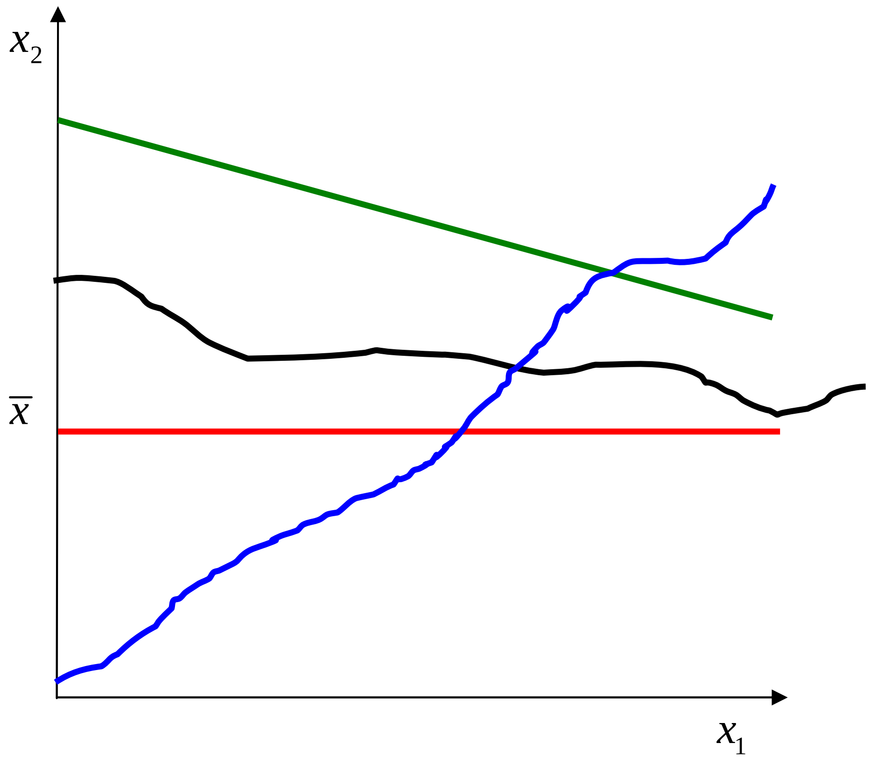
Stage 2: firm 2 chooses its output $x_2 \geq 0$ after observing the output chosen by firm 1.

Given that both players must have the same perception of the game not only does player 2 observes the choice made by player 1 but also player 1 knows that player 2 observes its choice. That is, there is perfect information and both players have the same perception of the game.

(Note: a game with two stages, i.e. a sequential game, in which the second mover does not observe the output chosen by the first mover –i.e. an imperfect information game– would be equivalent to a simultaneous game, such as the Cournot game).

The strategy spaces of the players are as follows:

- $x_1 \geq 0$: any non-negative quantity serves as a strategy for player 1; equivalently $x_1 \in [0, \infty)$.



- The description of the strategies for player 2 is more complex. Recall that a strategy is a complete description of what a player would do if he/she were called on to play at each one of his/her decision nodes, independently of whether they are attaniable or not given the current behavior of the other(s) player(s). In the Stackelberg game, each possible output of firm 1 generates a different decision node for firm 2. Therefore, firm 2's strategy is a function $x_2(x_1)$ which tells us how much firm 2 is going to produce for each possible production of firm 1.

(iii) *Backward induction. Subgame perfect equilibrium*

Although the game seems too complex to be solved, we know that in perfect information games without ties the backward induction criterion proposes a unique solution which coincides with the unique subgame perfect equilibrium. The proceedure is similar to that used with finite games in the previous chapter.

We start from the last subgames, that is, at stage 2.

Stage 2

We eliminate the non credible threats or dominated actions in each subgame. Given an output of firm 1 (a subgame) x_1 the only credible threat is for firm 2 to choose a profit maximizing output level:

$$\max_{x_2 \geq 0} \Pi_2(x_1, x_2) \equiv p(x_1 + x_2)x_2 - C_2(x_2)$$

$$\frac{\partial \Pi_2}{\partial x_2} = p(x_1 + x_2) + x_2 p'(x_1 + x_2) - C_2'(x_2) = 0 \;\; (1) \rightarrow \overline{f}_2(x_1)$$

$$\frac{\partial^2 \Pi_2}{\partial x_2^2} = 2p'(x_1 + x_2) + x_2 p''(x_1 + x_2) - C_2''(x_2) < 0$$

Taking into account the non-negativity constraint, $x_2 \geq 0$, we have:

$f_2(x_1) = \max\left\{ \overline{f}_2(x_1), 0 \right\} \rightarrow$ ***Firm 2's strategy in the subgame perfect equilibrium***.

In finite games, the proceedure continues by eliminating all the non credible threats and computing the reduced game. In the Stackelberg game eliminating all the incredible threats is equivalent to eliminating player 1's strategies other than $f_2(x_1) = \max\left\{ \overline{f}_2(x_1), 0 \right\}$.

Stage 1

Player 1 anticipates that firm 2 will behave at each subgame according to the strategy $f_2(x_1) = \max\left\{ \overline{f}_2(x_1), 0 \right\}$. Firm 1's profit function in reduced form is:

$\Pi_1(x_1, f_2(x_1)) \equiv p(x_1 + f_2(x_1))x_1 - C_1(x_1).$

Therefore, the problem for firm 1 becomes:

$$\max_{x_1 \geq 0} \Pi_1(x_1, f_2(x_1)) \equiv p(\underbrace{x_1 + f_2(x_1)}_{x})x_1 - C_1(x_1).$$

$$\frac{d\Pi_1}{dx_1} = p(x_1 + x_2) + x_1[1 + f_2^{'}(x_1)]p^{'}(x_1 + x_2) - C_1^{'}(x_1) = 0 \;\; (2) \rightarrow x_1^L$$

$$\frac{d^2\Pi_1}{dx_1^2} < 0$$

Therefore, the **subgame perfect equilibrium** is the strategy profile

$$(x_1^L, f_2(x_1)).$$

(iv) *Example: linear demand and constant marginal cost*.

Stage 2

We eliminate the non credible threats or dominated actions in each subgame. Given an output of firm 1 (a subgame) x_1 the only credible threat is for firm 2 to choose a profit maximizing output level:

$$\max_{x_2 \geq 0} \Pi_2(x_1, x_2) \equiv p(x_1 + x_2)x_2 - C_2(x_2) \equiv [a - b(x_1 + x_2)]x_2 - cx_2$$

$$\frac{\partial \Pi_2}{\partial x_2} = p(x_1 + x_2) + x_2 p^{'}(x_1 + x_2) - C_2^{'}(x_2) = 0 \;\; (1) \rightarrow \overline{f}_2(x_1) = \frac{a - c - bx_1}{2b}$$

$$\frac{\partial^2 \Pi_2}{\partial x_2^2} = 2p^{'}(x_1 + x_2) + x_2 p^{"}(x_1 + x_2) - C_2^{"}(x_2) = -2b < 0$$

Taking into account the non-negativity constraint, $x_2 \geq 0,$ we get:

$$f_2(x_1) = \max\left\{\overline{f}_2(x_1), 0\right\} = \max\left\{\frac{a - c - bx_1}{2b}, 0\right\}$$ ***Firm 2's strategy in the SPE***.

Stage 1

Player 1 anticipates that firm 2 will behave at each subgame according to the strategy $f_2(x_1) = \max\left\{ \bar{f}_2(x_1), 0 \right\} = \max\left\{ \frac{a-c-bx_1}{2b}, 0 \right\}$. Firm 1's profit function in reduced form is:

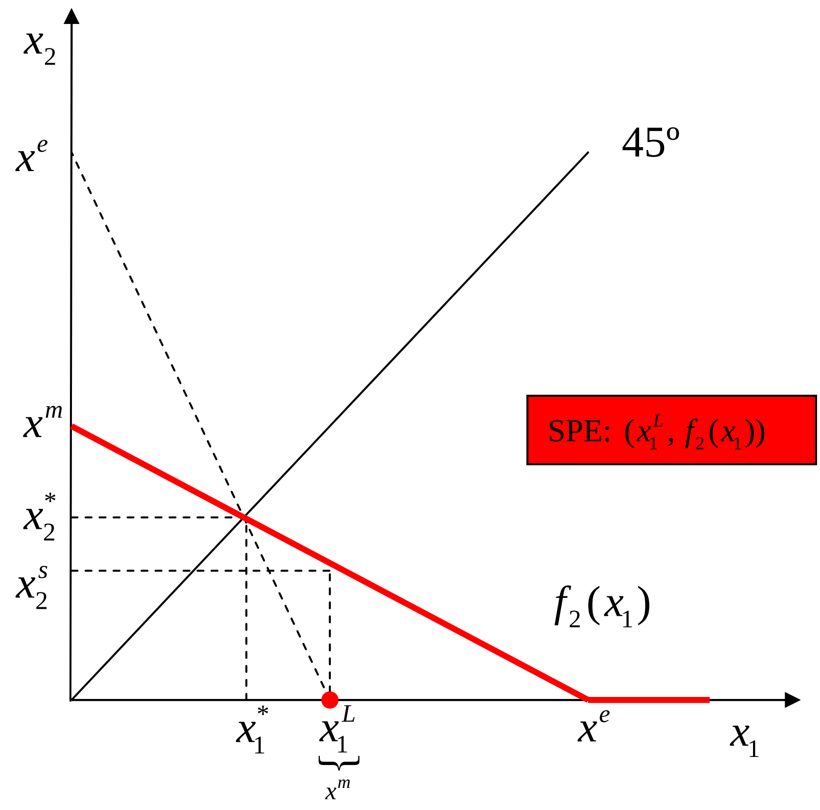
$\Pi_1(x_1, f_2(x_1)) \equiv p(x_1 + f_2(x_1))x_1 - C_1(x_1)$. Then firm 1's problem is:

$$\max_{x_1 \geq 0} \Pi_1(x_1, f_2(x_1)) \equiv [a-c-b(x_1 + f_2(x_1))]x_1 \equiv [a-c-b(x_1 + \frac{a-c-bx_1}{2b})]x_1 \equiv [\frac{a-c-bx_1}{2}]x_1$$

$$\frac{d\Pi_1}{dx_1} = p(x_1+x_2) + x_1[1+f_2^{'}(x_1)]p^{'}(x_1+x_2) - C_1^{'}(x_1) = a-c-2bx_1 = 0 \ \ (2) \rightarrow x_1^L = \frac{a-c}{2b}$$

$$\frac{d^2\Pi_1}{dx_1^2} < 0$$

Therefore, the **subgame perfect equilibrium is** $(x_1^L, f_2(x_1))$.



In order to obtain the profits of the firms we have to play the game.

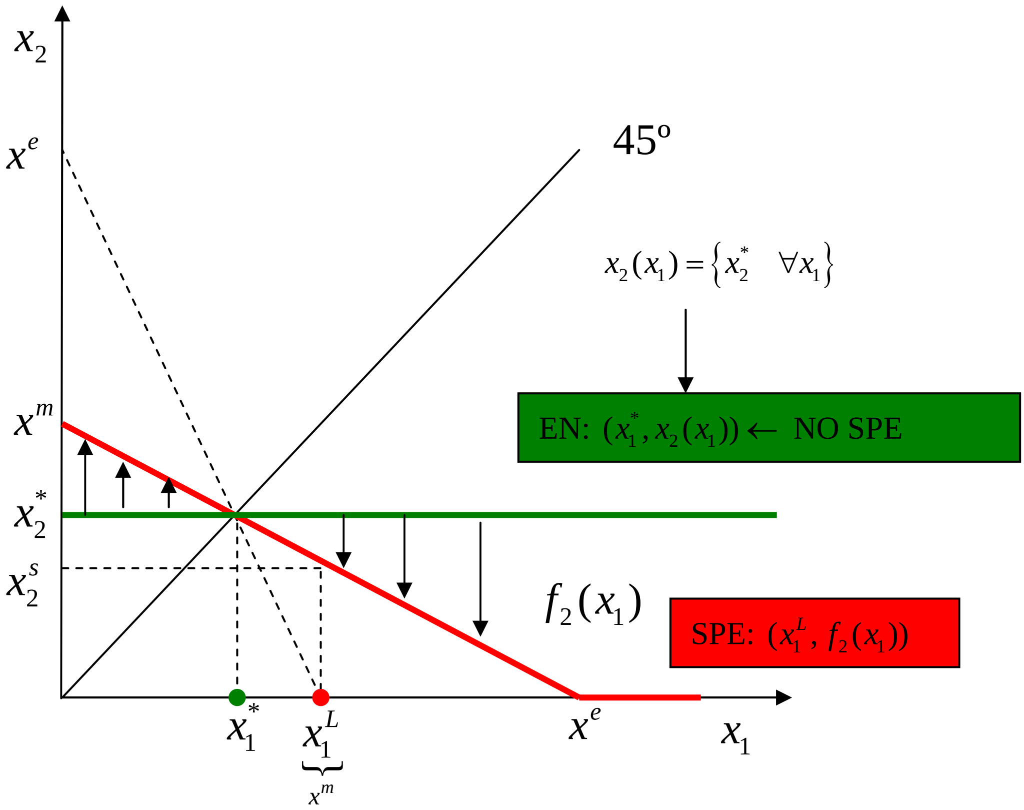
$$x_2^s = f_2(x_1^L) = \frac{a-c-bx_1^L}{2b} = \frac{a-c-b\left(\frac{a-c}{2b}\right)}{2b} = \frac{a-c}{4b}$$

$$x^s = x_1^L + x_2^s = \frac{a-c}{2b} + \frac{a-c}{4b} = \frac{3(a-c)}{4b}$$

$$p^s = p(x^s) = a - bx^s = a - b\frac{3(a-c)}{4b} = \frac{a+3c}{4}; \;\; p^s - c = \frac{a-c}{4}$$

$$\Pi_1^L = (p^s - c)x_1^L = \frac{(a-c)}{4}\frac{(a-c)}{2b} = \frac{(a-c)^2}{8b} =; \;\; \Pi_2^s = (p^s - c)x_2^s = \frac{(a-c)}{4}\frac{(a-c)}{4b} = \frac{(a-c)^2}{16b}.$$

(v) *Other Nash equilibria which are not subgame perfect*



1.4. **Collusion and stability of agreements**

1.4.1. ***Short-term collusion***

(i) Cournot model. The collusion agreement is not a short-term equilibrium.

(ii) Bertrand model. The collusion agreement is not a short-term equilibrium.

(i) *Cournot model. The collusion agreement is not a short-term equilibrium*

If firms colluded they would be interested in maximizing aggregate profits.

$$\max_{x_1,x_2} \Pi_1(x_1,x_2)+\Pi_2(x_1,x_2) \equiv p(x_1+x_2)x_1 - C_1(x_1) + p(x_1+x_2)x_2 - C_2(x_2)$$

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial x_1} &= p(x_1^m+x_2^m)+(x_1^m+x_2^m)p'(x_1^m+x_2^m)-C_1'(x_1^m)=0 \quad (1)\\ \frac{\partial \Pi}{\partial x_2} &= p(x_1^m+x_2^m)+(x_1^m+x_2^m)p'(x_1^m+x_2^m)-C_2'(x_2^m)=0 \quad (2)\end{aligned}\right\} MR_I = C_1' = C_2'$$

When marginal costs are constant and equal across firms conditions (1) and (2) are identical. The two-equation system would have infinite solutions: any pair of outputs such that $x_1 + x_2 = x^m$ would maximize the industry profit. For these cases we also refer to the symmetric collusion agreement where each firm produces a half of the monopoly output: $x_i^m = \frac{x^m}{2},\ i = 1,2.$

We now show that the collusion agreement (implicit of course) cannot be supported as an equilibrium when the game is played once. In other words, we demonstrate that the strategy profile (x_1^m, x_2^m) is not a Nash equilibrium in the Cournot game.

Given the strategy $x_j \geq 0$ the best response of firm *i* is to choose a strategy x_i such that:

$$\max_{x_i \geq 0} \Pi_i(x_i,x_j) \equiv p(x_i+x_j)x_i - C_i(x_i)$$

$$\frac{\partial \Pi_i}{\partial x_i} = p(x_i+x_j)+x_i p'(x_i+x_j)-C_i'(x_i)=0 \quad (1) \rightarrow \overline{f}_i(x_j)$$

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = 2p'(x_i+x_j)+x_i p''(x_i+x_j)-C_i''(x_i)<0$$

So the best response function is: $f_i(x_j) = \max\left\{\overline{f}_i(x_j), 0\right\}$.

To check that the combination of strategies (x_1^m, x_2^m) is not a Nash equilibrium we calculate the marginal profit for each firm:

$$\frac{\partial \Pi_i(x_i^m, x_j^m)}{\partial x_i} = p(x_i^m + x_j^m) + x_i^m p'(x_i^m + x_j^m) - C_i'(x_i^m) = -x_j^m \underbrace{p'(x_i^m + x_j^m)}_{<0} > 0$$

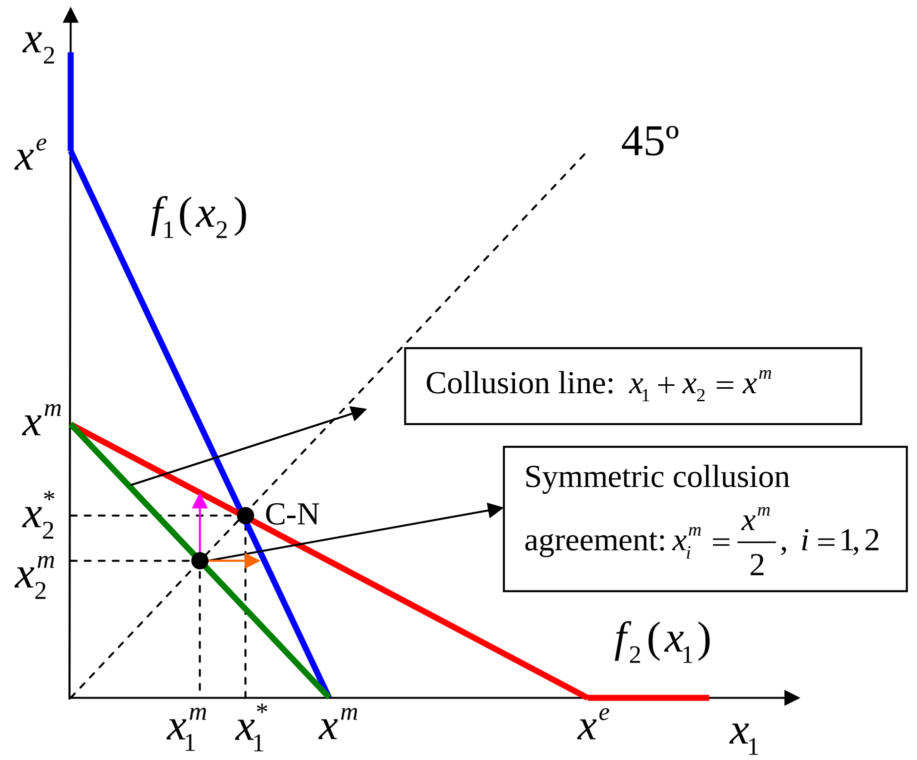
By definition of collusion agreement.

Then starting from the collusion agreement an increase in output leads to an increase in the firm *i*'s profit and, therefore, firm *i* would have an incentive to break the collusion agreement. Put differently, given the definition of best response function $\frac{\partial \Pi_i(f_i(x_j^m), x_j^m)}{\partial x_i} = 0$ and as $\frac{\partial \Pi_i(x_i^m, x_j^m)}{\partial x_i} > 0$ then $f_i(x_j^m) > x_i^m$.

As we know what the optimal deviation for firm *i* is if it decides to break the collusion agreement, $f_i(x_j^m)$, we denote by $\overline{\Pi}_i$ the profit that *i* would obtain if it deviates optimally and the other firms do not deviate. That is,

$$\overline{\Pi}_i = \Pi_i(f_i(x_j^m), x_j^m).$$

Graphic analysis: linear demand and constant marginal cost



Oligopoly

It is easy to generalize the above result to the case of *n* firms. The condition which defines the collusion agreement (the strategy profile maximizing the aggregate profit) is:

$$p(x_i^m + x_{-i}^m) + (x_i^m + x_{-i}^m)\, p'(x_i^m + x_{-i}^m) - C_i'(x_i^m) = 0 \quad i = 1,..,n.$$

To show that the strategy profile $(x_1^m,..,x_n^m)$ is not a Nash equilibrium we obtain the marginal profit of each firm:

$$\frac{\partial \Pi_i(x_i^m, x_{-i}^m)}{\partial x_i} = p(x_i^m + x_{-i}^m) + x_i^m\, p'(x_i^m + x_{-i}^m) - C_i'(x_i^m) = -x_{-i}^m \underbrace{p'(x_i^m + x_{-i}^m)}_{<0} > 0$$

By definition of the collusion agreement.

Then starting from the collusion agreement an increase in the output of firm *i* also increases its profit and, therefore, firm *I* would have an incentive to break the collusion agreement. In other words, given the definition of best response function $\frac{\partial \Pi_i(f_i(x_{-i}^m), x_{-i}^m)}{\partial x_i} = 0$ and as $\frac{\partial \Pi_i(x_i^m, x_{-i}^m)}{\partial x_i} > 0$ then $f_i(x_{-i}^m) > x_i^m$.

As we know what the optimal deviation for firm *i* is if it decides to break the collusion agreement, $f_i(x_{-i}^m)$, we denote by $\overline{\Pi}_i$ the profit that *i* would obtain if it deviates optimally and the other firms do not deviate. That is,

$$\overline{\Pi}_i = \Pi_i(f_i(x_{-i}^m), x_{-i}^m).$$

(ii) *Bertrand model. The collusion agreement is not a short-term equilibrium*

Consider the Bertrand model with homogeneous product and constant (and common) marginal cost. The strategy profile which represents the symmetric collusion agreement is (p^m, p^m). The profit of each firm is:

$$\Pi_i^m = \Pi_i(p^m, p^m) = (p^m - c)\frac{1}{2}D(p^m) = \frac{1}{2}\Pi^m$$

We know (it has been demonstrated) that a combination of strategies of the type $p_i = p_j > c$ is not a Nash equilibrium. Any firm would have an incentive to deviate unilaterally. For example, we can choose $p_i^{'} = p^m - \varepsilon$ (where ε is an arbitrary positive amount as small as necessary). Of course, there are infinite deviations such that firm *i* is better off.

Finding the optimal deviation for firm *i* is more problematical. The best that it can do is to undercut its rival's price by the lowest amount possible, $\varepsilon > 0, \varepsilon \to 0$. Although we do not have that optimal deviation well-defined we can be as near as we wish to monopoly price. Let $\overline{\Pi}_i$ be firm *i*'s profit when it optimally breaks the collusion agreement and the rival keeps it. That is,

$$\overline{\Pi}_i = \Pi_i(p^m - \varepsilon, p^m) = (p^m - \varepsilon - c)D(p^m - \varepsilon) \underbrace{\simeq}_{\varepsilon \to 0} (p^m - c)D(p^m) = \Pi^m$$

1.4.2. *Stability of agreements. Finite temporal horizon and infinite temporal horizon*

We have shown that in the short term the collusion (cooperation) agreement cannot hold as an equilibrium in either the Cournot game or the Bertrand game. In this section, we study the possibilities of collusion or cooperation when the game is repeated.

(i) ***Finite temporal horizon***

Backward induction argument $\rightarrow$ collusion (cooperation) cannot be supported as an equilibrium (at each stage firms behave as in the one-shot game). The reasoning is similar to that in the Prisoner's Dilemma.

(ii) ***Infinite temporal horizon***

There are two ways of interpreting an infinite temporal horizon:

(i) *Literal interpretation*: the game is repeated an infinite number of times. In this context, to compare two alternative strategies a player must compare the discounted present value of the respective gains. Let δ be the discount factor, $0 < \delta < 1$, and let r be the discount rate ($0 < r < \infty$) where $\delta = \frac{1}{1+r}$.

(ii) *Informational interpretation*: the game is repeated a finite but unkonwn number of times. At each stage, there is a probability $0 < \delta < 1$ of the game continuing. In this setting, each player must compare the expected value (which might be also discounted) of the different strategies.

We shall see that the existence of *implicit punishment threats* may serve to maintain collusion as an equilibrium of the repeated game.

Note first that there is a subgame perfect equilibrium of the infinitely repeated game where each player plays his/her short-term Nash equilibrium strategy in each period. In the Cournot model such a strategy would consist of "producing in each period the Cournot quantity independently of past history". In the Bertrand model that strategy would consist of "charging a price equals to marginal cost independently of past history".

We next study the possibility that there may be another subgame perfect equilibrium where players cooperate with each other. Consider the following combination of long term strategies: $s_i^c \equiv \{s_{it}(H_{t-1})\}_{t=1}^{\infty}$, $i = 1,2$.

where,

$$s_{it}^c(H_{t-1}) = \begin{cases} \overbrace{\text{"cooperate"}}^{\text{to collude}} \text{ if all elements of } H_{t-1} \text{ are equal to ("cooperate","cooperate") or } t=1 \\ \text{"not cooperate"(the short-term NE strategy)} \qquad \text{otherwise} \end{cases}$$

(in Cournot:

$$s_{it}^c(H_{t-1}) = \begin{cases} x_i^m & \text{if all elements of } H_{t-1} \text{ are equal to } (x_i^m, x_{-i}^m) \text{ or } t=1 \\ x_i^* & \text{otherwise} \end{cases}$$)

(in Bertrand:

$$s_{it}^c(H_{t-1}) = \begin{cases} p^m & \text{if all elements of } H_{t-1} \text{ are equal to } (p^m, p^m) \text{ or } t=1 \\ c & \text{otherwise} \end{cases}$$)

Note that these long term strategies incorporate "implicit punishment threats" in case of breach of the (implicit) cooperation agreement. The threat is credible because "confess" in each period (independently of the past history) is a Nash equilibrium of the repeated game.

To check whether it is possible in this context to maintain cooperation as an equilibrium, we must check that players have no incentive to deviate; that is, we must check that the combination of strategies (s_1^c, s_2^c) constitutes a Nash equilibrium of the repeated game.

Notation

$\Pi_i^m \rightarrow$ Firm *i*'s profit under collusion at each stage of the game.

$\Pi_i^* \rightarrow$ Firm *i*'s profit in the short-term Nash equilibrium at each stage of the game.

$\overline{\Pi}_i \rightarrow$ Firm *i*'s profit if the other firms cooperate and it optimally deviates.

$$\overline{\Pi}_i > \Pi_i^m > \Pi_i^*$$

The discounted present value for firm *i* in the strategy profile (s_1^c, s_2^c) is given by:

$$\pi_i(s_i^c, s_j^c) = \Pi_i^m + \delta\Pi_i^m + \delta^2\Pi_i^m + ... = \Pi_i^m(1 + \delta + \delta^2 + ...) = \frac{\Pi_i^m}{1-\delta}$$

If firm *i* deviates in the first period its gains are:

$$\pi_i(\overline{s}_i, s_j^c) = \overline{\Pi}_i + \delta\Pi_i^* + \delta^2\Pi_i^* + ... = \overline{\Pi}_i + \delta(1 + \delta + \delta^2 + ...)\Pi_i^* = \overline{\Pi}_i + \delta\frac{\Pi_i^*}{1-\delta}$$

Cooperation is supported as a Nash equilibrium if no player has any incentive to deviate; that is, when $\pi_i(s_i^c, s_j^c) \geq \pi_i(s_i, s_j^c)$. It is straightforward to check that if $\delta \geq \overline{\delta}$ no firm has an incentive to break the collusion agreement, where

$$\overline{\delta} = \frac{\overline{\Pi}_i - \Pi_i^m}{\overline{\Pi}_i - \Pi_i^*}.$$

Basic Bibliography

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