The Strategic Choice of Spatial Price Policy: Mixed Strategies Challenge the Classic Prisoner's Dilemma

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Abstract

In a seminal contribution, Thisse and Vives (1988) demonstrate that, in a two-stage pricing policy/price game, price discrimination emerges as a dominant strategy, trapping firms into a Prisoner's Dilemma, both in a linear city model with linear transport costs and in a circular market model with quadratic transport costs. Their result has profoundly shaped subsequent research in economics, management, and finance. Importantly, they make a specific modelling choice. Because the asymmetric subgames admit no pure-strategy Nash equilibrium under simultaneous price competition, they adopt the sequential Stackelberg standard: the mill-pricing firm becomes the price leader and the discriminatory firm responds optimally. Here, rather than modifying the original framework, I compute the unique mixed-strategy Nash equilibrium for asymmetric subgames. The main finding overturns the standard conclusion under maximum differentiation: the Prisoner's Dilemma vanishes, and the pricing-policy game becomes a coordination game, where mutual uniform pricing Pareto dominates mutual discrimination. In the linear-city framework, however, sufficiently large marginal-cost asymmetries restore price discrimination as a dominant strategy, so that the pricing-policy game may—or may not—exhibit the structure of a Prisoner's Dilemma.

JEL codes: D43, L13, R32.

Keywords: price discrimination, spatial competition, product differentiation, cost asymmetry, mixed-strategy Nash equilibrium, Prisoner's Dilemma.

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1 Introduction

This paper examines whether the celebrated Prisoner's Dilemma result of Thisse and Vives (1988) (henceforth TV) survives when their key modeling assumption is replaced by its natural game-theoretic alternative. TV consider a two-stage game where firms first simultaneously choose between uniform pricing (U) and discriminatory pricing (D), and then compete in prices—analyzing both a linear city model with linear transport costs and a circular market model with quadratic transport costs. In both models they show that price discrimination is a dominant strategy for each firm, generating a Prisoner's Dilemma: under discriminatory competition both firms earn less than under uniform mill pricing, yet each prefers to discriminate independently of the other's choice. Because the asymmetric subgame—in which one firm mills and the other discriminates—admits no pure-strategy Nash equilibrium under simultaneous competition, TV assume the mill-pricing firm acts as a Stackelberg leader.

In this paper, I replace the Stackelberg standard with the unique mixed-strategy Nash equilibrium (MSNE) of each asymmetric subgame. A key feature of this equilibrium, highlighted by Gabszewicz and Thisse (1992), is that price discrimination enlarges the strategy space relative to uniform pricing in the same way that mixed strategies enlarge it relative to pure strategies. This observation makes the structure of the MSNE transparent: in every asymmetric subgame the uniform-pricing firm randomizes over its mill price while the discriminating firm plays a pure delivered-price schedule. Because the mill pricer's price is no longer known, the discriminating firm can no longer position its schedule just below the rival's delivered price at every location; instead it must set a defensive schedule without knowing the rival's price. This substantially compresses the discriminating firm's expected profit and, for a wide range of parameters, overturns the Prisoner's Dilemma.

TV is widely cited across economics, finance, and management, in fields such as spatial competition, oligopolistic price discrimination and personalized pricing. Among many others, and without aiming to be exhaustive, see Tirole 1988; Anderson and de Palma 1988; Norman and Thisse 1996; Eber 1997; Laffont et al. 1998; Armstrong and Vickers 2001; Shaffer and Zhang 2002; Cambini and Valletti 2003; Liu and Serfes 2004; Degryse and Ongena 2005; Ellison 2005;Armstrong 2006; Chen 2006; Stole 2007; Galeotti and Moraga-González 2008; Ghose and Huang 2009; Fudenberg and Villas-Boas 2012; Matsumura and Matsushima 2015; Gill and Thanassoulis 2016; Adams and Williams 2019; Chen et al. 2022; Esteves 2022; Jullien et al. 2022; Houba et al. 2023; Assad et al. 2024; Lu and Matsushima 2024; Rhodes and Zhou 2024; Vives and Ye 2025; and Gokan et al. 2026. In both models the Prisoner's Dilemma result rests on what has become the "Stackelberg standard" for the asymmetric subgame in which one firm mills and the other discriminates. As TV acknowledge, no pure-strategy Nash equilibrium exists in that subgame under simultaneous competition, so they assume the mill-pricing firm moves first as price leader. It turns out that this assumption gives the discriminating follower an *exceptionally* high payoff that makes discrimination dominant.

Aguirre and Martín (2001) (AM) already question this modeling choice. Working within the linear-city model with linear transport costs and equal costs, they show that under simultaneous price competition the policy game has two Nash equilibria—mutual discrimination and mutual uniform pricing—so the Prisoner's Dilemma disappears. AM establish their result via an inequality, without deriving the exact discriminating-firm payoff.

Here, I study the relevance of this modelling choice in full generality and across both TV models. I derive the unique MSNE of every asymmetric subgame and obtain exact closed-form payoffs throughout. The contributions are as follows.

Linear-city model with asymmetric costs. Because the two firms have different marginal costs, the two asymmetric subgames have genuinely different MSNEs with different payoffs. I derive exact closed-form expressions for all eight payoffs and identify two thresholds that partition the parameter space into three regions: a coordination game (small cost asymmetry), a Prisoner's Dilemma (intermediate cost asymmetry), and a region where mutual discrimination is the unique Nash equilibrium without Prisoner's Dilemma tensions (large cost asymmetry). In the equal-cost case the formula supplies the exact discriminating-firm payoff that AM show lies below the uniform-pricing profit but do not compute.

Circular market with symmetric firms. I derive the unique MSNE of the asymmetric subgame in closed form. The discriminating firm's payoff falls strictly below the uniform-pricing payoff, so mutual uniform pricing is a Nash equilibrium alongside mutual discrimination, and the policy game is always a coordination game in this model.

A unifying result runs through both models: in every asymmetric subgame the mill-pricing firm earns the same payoff under Stackelberg and MSNE, while the entire difference between the two standards falls on the discriminating firm, whose payoff drops strictly when simultaneous competition replaces the Stackelberg standard.

Over the past few decades a large body of work on the strategic choice of price policy has been developed. This literature is in fact important and extensive, reaching several fields of research as spatial competition, oligopoly price discrimination and personalized pricing among others. Many researches have adopted the Stackelberg standard on the grounds that deriving the mixed-strategy Nash equilibrium of the asymmetric subgame is prohibitively complex. The present paper shows that price discrimination enlarges the strategy space in a similar way that mixed strategies do, and at equilibrium only the mill-pricing firm randomizes while the discriminating firm plays a pure schedule. This makes the structure of the MSNE transparent showing that the resolution is not intractable.

The paper is organized as follows. Section 2 treats the linear-city model with asymmetric costs. Section 3 studies the circular product differentiation model. Section 4 discusses the results jointly. Section 5 concludes.

2 The Spatial Competition Model with Asymmetric Costs

2.1 Model

I follow TV Section IIIA. Consumers are distributed uniformly on a segment of length 1, with firm 1 at location 0 and firm 2 at location 1. Transport cost from a firm located at a to a consumer located at x is a linear function of distance: $t(|x-a|)$, where $t>0$ is the transport rate. Following TV I assume asymmetric marginal costs: firm 1 has zero marginal cost $c_1 = 0$ and Firm 2 has marginal cost $c_2 = c \geq 0$. I set $\alpha = c/t \in [0,1)$; the condition $\alpha < 1$ guarantees that firm 2 sells a positive quantity in equilibrium. Firms are assumed to be risk neutral.

I consider a two-stage game. In Stage 1 each firm independently and simultaneously commits to either uniform pricing (U, a single mill price for all consumers) or discriminatory pricing (D, a delivered price that may vary by consumer location). In Stage 2 firms compete in prices simultaneously. We solve the game by backward induction.

Under uniform pricing a firm charges a mill price $p_i \geq c_i$ and consumers pay $p_i + t|x-i|$ for $i \in \{0,1\}$. Under discriminatory pricing a firm charges a delivered price at each location $p_i(x) \geq c_i + t|x-i|$.[1] Demand is unit-inelastic: each consumer demands one unit from the firm with the lowest final price, and the reservation price is assumed high enough that the entire market is covered in equilibrium.

2.2 Symmetric subgames

Lemma 1 (Mutual uniform pricing)**.** *Under (U,U), the unique Nash equilibrium has mill prices $p_1^* = t(3+\alpha)/3$ and $p_2^* = t(3+2\alpha)/3$, and equilibrium profits $\pi_1^{UU} = t(3+\alpha)^2/18$ and*

[1]It is standard in the literature on price competition to require that prices are above or equal to marginal cost, given that a price below marginal cost is (weakly) dominated. Under uniform pricing, this requires $p_i \geq c_i$. Under delivered pricing, it requires that the price at any location is at least marginal cost plus transport cost; a delivered price below this is a dominated strategy.

$\pi_2^{UU} = t(3-\alpha)^2/18$.

Proof. Residual demands of firm 1 and firm 2 are, respectively, $D_1(p_1, p_2) = 1/2 + (p_2 - p_1)/(2t)$ and $D_2(p_1, p_2) = 1/2 + (p_1 - p_2)/(2t)$. From the first-order conditions of the profit maximization problems, it is direct to obtain $p_1^* = t(3+\alpha)/3$ and $p_2^* = t(3+2\alpha)/3$. Equilibrium quantities of firm 1 and firm 2 are, respectively, $(3+\alpha)/6$ and $(3-\alpha)/6$. Hence $\pi_1^{UU} = t(3+\alpha)^2/18$ and $\pi_2^{UU} = t(3-\alpha)^2/18$. □

Remark 1. At $\alpha = 0$: $p_1^* = p_2^* = t$ and $\pi_1^{UU} = \pi_2^{UU} = t/2$—the standard Hotelling result with firms at the extremes of the market.

Lemma 2 (Mutual discriminatory pricing)**.** *Under (D, D): $p^*(x) = \max\{tx,\, c + t(1-x)\}$. Equilibrium profits are $\pi_1^{DD} = t(1+\alpha)^2/4$ and $\pi_2^{DD} = t(1-\alpha)^2/4$.*

Proof. Following Lederer and Hurter (1986) and TV, at each location the equilibrium price is the maximum of the total marginal cost.[2] That is, $p^*(x) = \max\{tx,\, c + t(1-x)\}$. The market boundary is $x^{DD} = (1+\alpha)/2$. Equilibrium profits are

$$\pi_1^{DD} = \int_0^{(1+\alpha)/2} \big[p^*(x) - tx\big]\, dx = \int_0^{(1+\alpha)/2} \big[c + t(1-2x)\big]\, dx = \frac{t(1+\alpha)^2}{4},$$
$$\pi_2^{DD} = \int_{(1+\alpha)/2}^1 \big[p^*(x) - c - t(1-x)\big]\, dx = \int_{(1+\alpha)/2}^1 \big[t(2x-1) - c\big]\, dx = \frac{t(1-\alpha)^2}{4}. \qquad \square$$

□

Remark 2. It is satisfied that $\pi_2^{UU} > \pi_2^{DD}$ for all $\alpha \in [0, 1)$, so Firm 2 always prefers (U, U) to (D, D). $\pi_1^{UU} > \pi_1^{DD}$ if and only if $\alpha < \alpha^* \approx 0.784$. Hence (D, D) is Pareto-inefficient if $\alpha < \alpha^*$. When $\alpha > \alpha^*$ then $\pi_1^{UU} < \pi_1^{DD}$ and $\pi_2^{UU} > \pi_2^{DD}$. When $\alpha = 0$ then $\pi_1^{UU} = \pi_2^{UU} = \pi^{UU} = t/2 > t/4 = \pi_1^{DD} = \pi_2^{DD} = \pi^{DD}$; that is, $\pi^{UU} = 2\pi^{DD}$.

2.3 The asymmetric (U, D) subgame

Firm 1 sets mill price $p_1 \geq 0$; firm 2 sets delivered-price schedule $p_2(x) \geq c + t(1-x)$ for all x. Firm 1 captures $[0, \tilde{x}]$ where $p_1 + t\tilde{x} = p_2(\tilde{x})$.

As TV note, no pure-strategy Nash equilibrium exists in this subgame under simultaneous price competition.

[2] At each location price competition is like Bertrand with asymmetric marginal costs. To avoid an ε-equilibrium, we follow Lederer and Hurter (1986) in assuming an efficient sharing rule: when prices are equal, consumers buy from the more efficient firm (the one with the lower delivered cost).

Solution under the Stackelberg standard. Under TV's Stackelberg standard firm 1 is the price leader and sets p_1. Firm 2 best-responds by setting $p_2(x) = p_1 + tx$ wherever this exceeds its cost constraint $c + t(1-x)$, i.e. for $x \geq \tilde{x} = (c + t - p_1)/(2t)$. The price that maximizes firm 1's profit is $p_1^* = t(1+\alpha)/2$, and the profits for the two firms are:

$$\pi_1^{UD} = \frac{t(1+\alpha)^2}{8}, \qquad \pi_2^{UD} = \frac{t(3-\alpha)^2}{16}.$$

Solution under simultaneous competition (MSNE). Under simultaneous competition firm 1 randomizes while firm 2 follows a pure strategy. I derive the unique MSNE following four steps: (1) I obtain the type of firm 2's pricing policy that leaves firm 1 indifferent among prices in an interval, determining the support of the cumulative distribution function. (2) Next firm 2's expected profit is obtained. (3) The first-order condition of its profit maximization yields to an ordinary differential equation (ODE). (4) By solving it, I derive firm 1's cumulative distribution function (CDF).

Step 1: The support $[p_L, p_H]$.

Upper boundary. For firm 1 to be indifferent across its support, its per-unit profit $p_1 \cdot \tilde{x}(p_1) = \bar{\pi}$ must be constant for some $\bar{\pi} > 0$, giving territory $\tilde{x}(p_1) = \bar{\pi}/p_1$ and the indifference schedule $p_2^*(x) = \bar{\pi}/x + tx$. This schedule must satisfy firm 2's constraint $p_2^*(x) \geq c + t(1-x)$, i.e. $\bar{\pi} \geq x[(c+t) - 2tx]$.

Lemma 3 (Feasibility floor, (U, D))**.** *The constraint $p_2^*(x) \geq c + t(1-x)$ requires $\bar{\pi} \geq t(1+\alpha)^2/8$, binding at $x^* = (1+\alpha)/4$.*

Proof. $x[(c+t) - 2tx]$ is maximized at $x^* = (c+t)/(4t) = (1+\alpha)/4$ with value $(c+t)^2/(8t) = t(1+\alpha)^2/8$. □

Setting $\bar{\pi} = t(1+\alpha)^2/8$, firm 1's guaranteed payoff in any MSNE is $\bar{\Pi}_1 = \bar{\pi} = t(1+\alpha)^2/8$. But this is precisely the payoff firm 1 obtains by setting the Stackelberg price $p_1^S = t(1+\alpha)/2$, since under Stackelberg firm 2's best response gives firm 1 territory $\tilde{x}(p_1^S) = \bar{\pi}/p_1^S = (1+\alpha)/4$ and profit $p_1^S \cdot (1+\alpha)/4 = t(1+\alpha)^2/8 = \bar{\pi}$. Firm 1 can always guarantee this payoff by deviating to p_1^S. Therefore no mill price above $p_1^S = t(1+\alpha)/2$ can be in the support, and we set

$$p_H = p_1^S = \frac{t(1+\alpha)}{2}.$$

Firm 2's profit at the boundary location $x = \bar{\pi}/p$ is $\pi_2^x(p) = [2p - t(1+\alpha)]^2/(4p)$, which equals zero at $p = p_H$, consistent with p_H being the upper boundary. We verify below (Step 4) that the cumulative equilibrium function (CDF) satisfies $F_1(p_H) = 1$.

Lower boundary. Before deriving p_L I establish that firm 2 can guarantee its (D, D) payoff regardless of p_1; this guarantee is what prevents firm 1 from pricing below p_L.

Remark 3 (Guaranteed profit floor, (U, D)). Independently of firm 1's mill price, firm 2 can always guarantee at least $\pi_2^{DD} = t(1-\alpha)^2/4$ by playing the Lederer–Hurter schedule $p_2(x) = tx$ on $[x^{DD}, 1]$ (pricing at firm 1's total delivered cost). Firm 2's profit at $x \geq x^{DD}$ is $tx - c - t(1-x) = 2tx - c - t \geq 0$. This strategy satisfies the constraint $p_2(x) \geq c + t(1-x)$ since $tx \geq c + t(1-x)$ iff $x \geq x^{DD}$. It yields π_2^{DD} regardless of p_1, so any MSNE must deliver firm 2 at least π_2^{DD}.

At mill price p_1, firm 1 captures $[0, \tilde{x}(p_1)]$ with $\tilde{x}(p_1) = \bar{\pi}/p_1$. I require that at the lowest price p_L in the support, firm 1's territory reaches exactly the natural (D, D) boundary: $\tilde{x}(p_L) = x^{DD} = (1+\alpha)/2$. If firm 1 priced below p_L it would capture beyond x^{DD}, but firm 2 can always guarantee π_2^{DD} there (Remark 3), so such a deviation cannot be part of an equilibrium. Solving:

$$\frac{\bar{\pi}}{p_L} = \frac{1+\alpha}{2} \implies p_L = \frac{t(1+\alpha)^2/8}{(1+\alpha)/2} = \frac{t(1+\alpha)}{4}.$$

This also satisfies firm 1's constraint $p_L \geq 0$. The support is therefore $[p_L, p_H] = [t(1+\alpha)/4,\, t(1+\alpha)/2]$.

Proposition 1 (firm 1's MSNE payoff, (U, D))**.** *In the (U, D) MSNE, firm 1 earns $\bar{\Pi}_1 = \bar{\pi} = t(1+\alpha)^2/8$ for every $p_1 \in [p_L, p_H]$. This equals firm 1's Stackelberg payoff.*

Step 2: firm 2's expected profit. Given F_1, the cumulative distribution function (CDF) of firm 1's mill price, at each location x firm 2 sets $p_2(x)$ to maximize its expected profit. Firm 2 captures x iff $p_2(x) < p_1 + tx$, i.e. iff p_1 exceeds the reserve price $R(x) \equiv p_2(x) - tx$, which is the mill price at which firm 1's delivered price at x exactly equals $p_2(x)$. The capture probability is therefore $\Pr[p_1 > R(x)] = 1 - F_1(R(x))$. The profit at location x—the difference between the delivered price and firm 2's total cost at x—is $p_2(x) - c - t(1-x)$, so the expected profit at x is

$$\pi_2(x) = \big[p_2(x) - c - t(1-x)\big] \cdot \big[1 - F_1\big(R(x)\big)\big].$$

Step 3: First-order condition. Differentiating $\pi_2(x)$ with respect to $p_2(x)$, or equivalently with respect to $R \equiv p_2(x) - tx$, and setting equal to zero:

$$\big[1 - F_1(R)\big] = \pi_2^x(R) \cdot f_1(R),$$

where $f_1 = F_1'$ is the density of firm 1's mixing distribution and $\pi_2^x(R) = p_2(x) - c - t(1-x) = R + 2tx - t(1+\alpha)$ is firm 2's profit at location x. At the equilibrium reserve price $R^*(x) = \bar{\pi}/x$, substituting $x = \bar{\pi}/R^*$ gives $\pi_2^x(R^*) = [2R^* - t(1+\alpha)]^2/(4R^*)$ derived in Step 1. This gives the ODE:

$$\frac{d\ln(1-F_1)}{dp_1} = -\frac{4p_1}{\big[2p_1 - t(1+\alpha)\big]^2}. \tag{1}$$

Step 4: CDF. Integrating the ODE and setting $\varepsilon = p_H - p_1 = t(1+\alpha)/2 - p_1$ yields

$$\frac{d\ln(1-F_1)}{d\varepsilon} = \frac{t(1+\alpha)}{2\varepsilon^2} - \frac{1}{\varepsilon},$$

which integrates to $\ln(1-F_1) = -t(1+\alpha)/(2\varepsilon) - \ln\varepsilon + C$, i.e. $1 - F_1 = K\exp(-t(1+\alpha)/(2\varepsilon))/\varepsilon$. The boundary condition $F_1(p_L) = 0$ at $\varepsilon_L = p_H - p_L = t(1+\alpha)/4$ gives $K = t(1+\alpha)/4 \cdot e^2$. The exponent $t(1+\alpha)/(2\varepsilon_L) = 2$ holds independently of α.

Verification. As $p_1 \to p_H$, i.e. $\varepsilon \to 0$, the solution satisfies $1 - F_1 = K\varepsilon^{-1}\exp(-(1+\alpha)t/(2\varepsilon)) \to 0$ since the exponential decay dominates the ε^{-1} factor. Hence $F_1(p_H) = 1$: the mill pricer is never indifferent between p_H and higher prices, confirming $p_H = t(1+\alpha)/2$ is indeed the upper boundary of the support.

Proposition 2 (Unique MSNE, (U, D))**.** *The unique mixing distribution for firm 1 has support $[t(1+\alpha)/4,\, t(1+\alpha)/2]$ with CDF*

$$F_1(p_1) = 1 - \frac{t(1+\alpha)}{4}\, e^2 \cdot \frac{\exp\big(-t(1+\alpha)/(2\varepsilon)\big)}{\varepsilon}, \quad \varepsilon = \tfrac{t(1+\alpha)}{2} - p_1.$$

The equilibrium schedule has two parts, where $R^*(x) \equiv p_2^*(x) - tx$ denotes the equilibrium reserve price:

- *$x \in [(1+\alpha)/4,\, (1+\alpha)/2]$: $p_2^*(x) = \bar{\pi}/x + tx$, so $R^*(x) = \bar{\pi}/x \in [p_L, p_H]$ (probabilistic capture);*
- *$x \in ((1+\alpha)/2,\, 1]$: $p_2^*(x) = t(1+\alpha)/4 + tx$, so $R^*(x) = t(1+\alpha)/4 = p_L$ (certain capture, $F_1(p_L) = 0$).*

Given F_1, firm 2's best response is unique (strict quasi-concavity at each x), so the MSNE is unique.

Proposition 3 (firm 2's expected payoff, (U, D))**.**

$$\pi_2^{UD} = \frac{t(1+\alpha)^2 k}{16} + \frac{t(1-\alpha)(3-\alpha)}{8},$$

where $E_3(x) = \int_1^\infty e^{-xs}\, s^{-3}\, ds$ *is the exponential integral of order* 3 *and* $k = e\, E_3(1) \approx 0.298$.[3]

Proof. By the two-part structure of Proposition 2:

Certain part ($x \in [(1+\alpha)/2, 1]$): profit at location x is $p_2^*(x) - c - t(1-x) = t(1+\alpha)/4 + tx - \alpha t - t + tx = t(-3 - 3\alpha + 8x)/4$. Integrating over $[(1+\alpha)/2, 1]$:

$$\int_{(1+\alpha)/2}^{1} \frac{t(-3-3\alpha+8x)}{4}\, dx = \frac{t(1-\alpha)(3-\alpha)}{8}.$$

Probabilistic part ($x \in [(1+\alpha)/4, (1+\alpha)/2]$): substituting $p = \bar{\pi}/x$ converts the integrand to $(2p - t(1+\alpha))^2/(4p)$. Setting $u = t(1+\alpha)/(2\varepsilon)$ with lower limit $u_L = 2$ (since $\varepsilon_L = t(1+\alpha)/4$), then $w = u - 1$, yields $t(1+\alpha)^2 k/16$ via $\int_1^\infty e^{-w} w^{-3}\, dw = E_3(1)$. $\square$

Figure 1 illustrates the CDF and density of the mill-pricing firm's mixing distribution at $\alpha = 0$. The density is hump-shaped, peaking near $p_1 \approx 0.354$, and falls to zero toward both boundaries. The mill pricer avoids $p_H = t/2$ — the Stackelberg price, which the discriminator could exploit if it were known — and also avoids $p_L = t/4$, which would surrender too much market share. Randomizing across the interior prevents the discriminating firm from conditioning its schedule on a known price.

2.4 The asymmetric (D, U) subgame

Now firm 2 sets mill price $p_2 \geq c$; firm 1 sets schedule $p_1(x) \geq tx$. Firm 1 captures x iff $p_1(x) < p_2 + t(1-x)$, equivalently iff p_2 exceeds the reserve price $R(x) \equiv p_1(x) - t(1-x)$, which is the mill price at which Firm 2's delivered price at x exactly equals $p_1(x)$. Note that $R(x)$ is now defined in terms of firm 1's schedule (not firm 2's as in Section 2.3). The cost asymmetry changes the structure qualitatively: firm 2 (the higher-cost firm) now holds the mill price.

Under TV's Stackelberg standard (firm 2 is leader, sets $p_2 \geq c$), Firm 1 best-responds by undercutting firm 2's delivered price wherever profitable ($x < (p_2 + t)/(2t)$). Firm 2 maximizes $(p_2 - c)(1 - \tilde{x})$, giving $p_2^* = t(1+\alpha)/2 \geq c$ and

$$\pi_2^{DU} = \frac{t(1-\alpha)^2}{8}, \qquad \pi_1^{DU} = \frac{t(3+\alpha)^2}{16}.$$

[3] Exponential integrals of this type are not unusual in economic analysis. They arise, for instance, in auction theory when computing expected revenues under optimal mechanisms with specific valuation distributions, and in search-and-matching models when deriving closed-form expressions for expected durations or surplus. The appearance of $E_3(1)$ here reflects the double-root structure of the ODE, which forces a particular substitution that reduces the payoff integral to this standard special function.

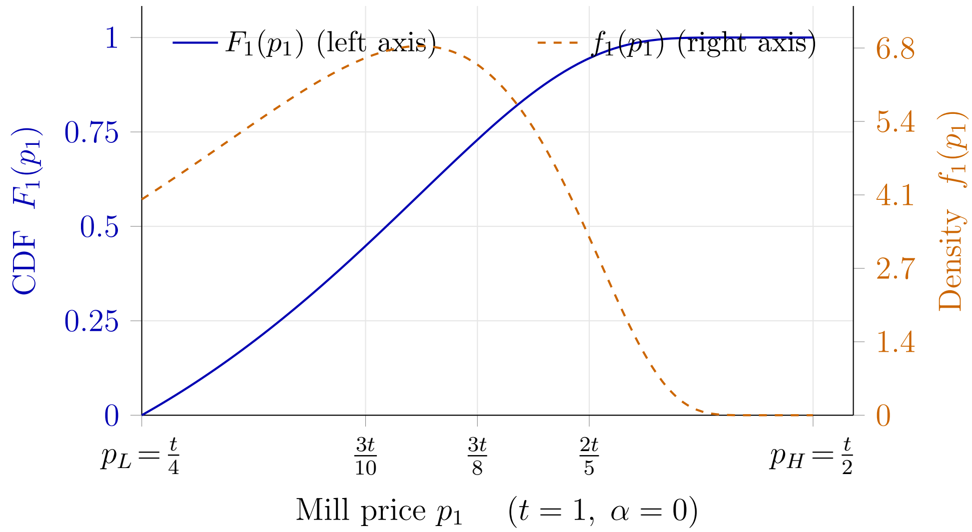

Figure 1: CDF $F_1(p_1)$ (solid blue, left axis) and density $f_1(p_1)$ (dashed orange, right axis) of the mill-pricing firm's mixing distribution in the (U, D) subgame, at $\alpha = 0$, $t = 1$. The support is $[p_L, p_H] = [t/4, t/2] = [0.25, 0.50]$. The density peaks at $p_1 \approx 0.354$ and falls sharply toward both boundaries: mass is concentrated in the interior of the support, and the mill pricer essentially never prices near $p_H = t/2$ or near $p_L = t/4$.

Under simultaneous competition, firm 2 randomizes; I derive the unique MSNE following the same four steps as in Section 2.3.

Step 1: The support $[p_L, p_H]$**.**

Upper boundary. For firm 2 to be indifferent, its per-unit profit $(p_2 - c)(1 - \tilde{x}) = \bar{\pi}_2$ must be constant, giving $\tilde{x} = 1 - \bar{\pi}_2/(p_2 - c)$. At the boundary $\tilde{x}$, one has $R^*(\tilde{x}) = p_2$ (the reserve price equals the mill price), so the equilibrium reserve price schedule is $R^*(x) = c + \bar{\pi}_2/(1-x)$, giving $p_1^*(x) = R^*(x) + t(1-x) = t(1-x) + c + \bar{\pi}_2/(1-x)$. This must satisfy Firm 1's constraint $p_1^*(x) \geq tx$.

Lemma 4 (Feasibility floor, (D, U))**.** *The constraint $p_1^*(x) \geq tx$ requires $\bar{\pi}_2 \geq t(1-\alpha)^2/8$, binding at $x^* = (3+\alpha)/4$.*

Proof. $p_1^*(x) \geq tx$ iff $\bar{\pi}_2/(1-x) \geq t(2x-1) - c$ for $x \in [(1+\alpha)/2, 1]$. The function $(1-x)(t(2x-1)-c)$ is maximized at $x^* = (3+\alpha)/4$ with value $t(1-\alpha)^2/8$. □

Setting $\bar{\pi}_2 = t(1-\alpha)^2/8$, firm 2's guaranteed payoff in any MSNE is $\bar{\Pi}_2 = \bar{\pi}_2 = t(1-\alpha)^2/8$. This is precisely the payoff firm 2 obtains by setting the Stackelberg price $p_2^S = t(1+\alpha)/2$: firm 1's best response gives firm 2 territory $[(3+\alpha)/4, 1]$ and profit $(p_2^S - c)(1 - (3+\alpha)/4) = t(1-\alpha)/2 \cdot (1-\alpha)/4 = t(1-\alpha)^2/8 = \bar{\pi}_2$. Firm 2 can always guarantee this payoff by deviating to p_2^S. Therefore no mill price above $p_2^S = t(1+\alpha)/2$ can be in the support, and we set

$$p_H = p_2^S = \frac{t(1+\alpha)}{2}.$$

Firm 1's profit at the boundary is $\pi_1^x(R) = [R - t(1+\alpha)/2]^2/(R-c)$, which equals zero at $R = p_H$, consistent with p_H being the upper boundary. We verify below (Step 4) that $F_2(p_H) = 1$.

Lower boundary. Before deriving p_L I establish that firm 1 can guarantee its (D, D) payoff regardless of p_2; this guarantee is what prevents firm 2 from pricing below p_L.

Remark 4 (Guaranteed profit floor, (D, U)). Independently of firm 2's mill price $p_2 \geq c$, firm 1 can always guarantee at least $\pi_1^{DD} = t(1+\alpha)^2/4$ by playing the Lederer–Hurter schedule $p_1(x) = c + t(1-x)$ for $x \in [0, x^{DD}]$ (pricing at firm 2's total delivered cost). For any $p_2 \geq c$, firm 1's price satisfies $p_1(x) = c + t(1-x) \leq p_2 + t(1-x)$, so firm 1 captures $[0, x^{DD}]$ with certainty. Firm 1's profit at each x is $c + t(1-x) - tx = c + t - 2tx > 0$ for $x < x^{DD}$; the constraint $p_1(x) \geq tx$ holds since $c + t - 2tx \geq 0$ throughout $[0, x^{DD}]$; and integrating gives $\pi_1^{DD} = t(1+\alpha)^2/4$ regardless of p_2. Therefore any MSNE must deliver firm 1 at least π_1^{DD}.

At mill price p_2, firm 2's market share is $[\tilde{x}(p_2), 1]$ with $\tilde{x}(p_2) = 1 - \bar{\pi}_2/(p_2 - c)$. I require that at the lowest price p_L in the support, firm 2's share starts at exactly the natural (D, D) boundary: $\tilde{x}(p_L) = x^{DD} = (1+\alpha)/2$. If firm 2 priced below p_L it would capture beyond x^{DD} toward firm 1's natural market share, but firm 1 can always guarantee π_1^{DD} there (Remark 4). Solving:

$$1 - \frac{\bar{\pi}_2}{p_L - c} = \frac{1+\alpha}{2} \implies p_L - c = \frac{t(1-\alpha)^2/8}{(1-\alpha)/2} = \frac{t(1-\alpha)}{4},$$

giving $p_L = c + t(1-\alpha)/4 = t(3\alpha+1)/4 \geq c$ for all $\alpha \in [0, 1)$. The support is therefore $[p_L, p_H] = [t(3\alpha+1)/4,\, t(1+\alpha)/2]$.

Proposition 4 (firm 2's MSNE payoff, (D, U))**.** *In the (D, U) MSNE, firm 2 earns $\bar{\Pi}_2 = \bar{\pi}_2 = t(1-\alpha)^2/8$ for every $p_2 \in [p_L, p_H]$. This equals firm 2's Stackelberg payoff.*

Remark 5 (Guaranteed profit floors: symmetry between subgames). In both asymmetric subgames the discriminating firm can guarantee its (D, D) payoff regardless of the mill pricer's strategy. In (U, D), firm 2 (discriminator) guarantees π_2^{DD} by playing $p_2(x) = tx$ on $[x^{DD}, 1]$ (Remark 3). In (D, U), firm 1 (discriminator) guarantees π_1^{DD} by playing $p_1(x) = c + t(1-x)$ on $[0, x^{DD}]$ (Remark 4). The mill-pricing firm has no analogous guarantee: both $\bar{\Pi}_1$ and $\bar{\Pi}_2$ fall short of the respective (D, D) payoffs.

Step 2: firm 1's expected profit. Given F_2, the CDF of firm 2's mill price, at each location x firm 1 sets $p_1(x)$ to maximize its expected profit. Firm 1 captures x iff $p_2 >$

$R(x) \equiv p_1(x) - t(1-x)$ (the reserve price defined above), i.e. with probability $1 - F_2(R(x))$. The profit at location x—delivered price minus firm 1's transport cost—is $p_1(x) - tx$, so the expected profit at x is

$$\pi_1(x) = \big[p_1(x) - tx\big] \cdot \big[1 - F_2\big(R(x)\big)\big].$$

Step 3: First-order condition. Differentiating $\pi_1(x)$ with respect to $p_1(x)$, or equivalently with respect to $R \equiv p_1(x) - t(1-x)$, and setting equal to zero:

$$\big[1 - F_2(R)\big] = \pi_1^x(R) \cdot f_2(R),$$

where $f_2 = F_2'$ is the density of firm 2's mixing distribution and $\pi_1^x(R) = p_1(x) - tx$ is firm 1's profit at location x. At the equilibrium reserve price $R^*(x) = c + \bar{\pi}_2/(1-x)$, substituting $x = 1 - \bar{\pi}_2/(R-c)$ gives

$$\pi_1^x(R^*) = R^* - t + \frac{2t\bar{\pi}_2}{R^* - c} = \frac{\big[R^* - t(1+\alpha)/2\big]^2}{R^* - c}.$$

This gives the ODE

$$\frac{d\ln(1-F_2)}{dp_2} = -\frac{p_2 - c}{\big[p_2 - t(1+\alpha)/2\big]^2}. \tag{2}$$

4

Step 4: CDF. Setting $\varepsilon = p_H - p_2 = (1+\alpha)t/2 - p_2$ and noting $p_2 - c = t(1-\alpha)/2 - \varepsilon$, the ODE becomes

$$\frac{d\ln(1-F_2)}{d\varepsilon} = \frac{t(1-\alpha)}{2\varepsilon^2} - \frac{1}{\varepsilon},$$

which integrates to $1 - F_2 = K\exp(-t(1-\alpha)/(2\varepsilon))/\varepsilon$. The boundary condition $F_2(p_L) = 0$ at $\varepsilon_L = p_H - p_L = t(1-\alpha)/4$ gives $K = t(1-\alpha)/4 \cdot e^2$. The exponent $(1-\alpha)t/(2\varepsilon_L) = 2$.

Verification. As $p_2 \to p_H$, i.e. $\varepsilon \to 0$, $1 - F_2 = K\varepsilon^{-1}\exp(-t(1-\alpha)/(2\varepsilon)) \to 0$, so $F_2(p_H) = 1$: the mill pricer is never indifferent between p_H and higher prices, confirming $p_H = t(1+\alpha)/2$ is the upper boundary of the support.

Proposition 5 (Unique MSNE, (D,U))**.** *The unique mixing distribution for firm 2 has support* $[t(3\alpha+1)/4,\ t(1+\alpha)/2]$ *with CDF*

$$F_2(p_2) = 1 - \frac{t(1-\alpha)}{4}\,e^2 \cdot \frac{\exp\big(-t(1-\alpha)/(2\varepsilon)\big)}{\varepsilon}, \quad \varepsilon = \tfrac{t(1+\alpha)}{2} - p_2,$$

[4] As in equation (1), the denominator is a perfect square, here with a double root at $p_H = t(1+\alpha)/2$, again arising from the feasibility floor driving the discriminant of the net-revenue quadratic to zero.

with exponent $t(1-\alpha)/(2\varepsilon_L) = 2$ *at* p_L. *The equilibrium schedule has two parts, where* $R^*(x) \equiv p_1^*(x) - t(1-x)$ *denotes the equilibrium reserve price:*

- $x \in [(1+\alpha)/2,\ (3+\alpha)/4]$*:* $R^*(x) = c + \bar{\pi}_2/(1-x) \in [p_L, p_H]$ *(probabilistic capture);*
- $x \in [0,\ (1+\alpha)/2]$*:* $p_1^*(x) = t(3\alpha+1)/4 + t(1-x)$*, so* $R^*(x) = t(3\alpha+1)/4 = p_L$ *(certain capture,* $F_2(p_L) = 0$*).*

Given F_2*, firm 1's best response is unique (strict quasi-concavity at each* x*), so the MSNE is unique.*

Remark 6 (Structural symmetry between subgames)*.* Both ODEs (1) and (2) yield a survival function of the form $G(p) = A\exp(-B/\varepsilon)/\varepsilon$ with exponent $B/\varepsilon_L = 2$ at p_L. The constant B equals $t(1+\alpha)$ in (U,D) and $t(1-\alpha)$ in (D,U): the entire asymmetry between the two subgames is encoded in the replacement of $(1+\alpha)$ by $(1-\alpha)$. This symmetry traces directly to the cost constraints: the feasibility floor in (U,D) binds at firm 2's full cost $c + t(1-x)$, yielding $\bar{\pi} = (c+t)^2/(8t)$; in (D,U) it binds at Firm 1's full cost tx, yielding $\bar{\pi}_2 = (t-c)^2/(8t)$. The replacements $c+t \to t-c$ and $t-c \to c+t$ are exact duals.

Proposition 6 (firm 1's expected payoff, (D,U))**.**

$$\pi_1^{DU} = \frac{t(1-\alpha)^2 k}{16} + \frac{t(1+\alpha)(3+\alpha)}{8}.$$

Proof. Certain part ($x \in [0,(1+\alpha)/2]$): firm 1 always captures this interval for any $p_2 \in [p_L, p_H]$, because its schedule $p_1^*(x) = p_L + t(1-x)$ prices strictly below firm 2's lowest possible delivered price $p_L + t(1-x)$, and $p_1^*(x) \geq tx$ holds throughout (the net $p_L + t - 2tx$ is minimal at $x = (1+\alpha)/2$ where it equals $t(1-\alpha)/4 > 0$). Integrating the profit at each location:

$$\int_0^{(1+\alpha)/2} \left[\frac{(3\alpha+1)t}{4} + t - 2tx\right] dx = \frac{t(1+\alpha)(3+\alpha)}{8}.$$

Probabilistic part ($x \in [(1+\alpha)/2,(3+\alpha)/4]$): substituting $q = \bar{\pi}_2/(1-x)$ converts the profit function to $(q - t(1-\alpha)/2)^2/q$; then $v = (1-\alpha)t/(2(t(1-\alpha)/2 - q))$ with lower limit $v_L = 2$ and $w = v - 1$ yields $t(1-\alpha)^2 k/16$. □

Remark 7*.* The equilibrium profit exceeds the guaranteed floor: $\pi_1^{DU} - \pi_1^{DD} = t(1-\alpha)^2 k/16 + (1-\alpha^2)t/8 > 0$ for all $\alpha \in [0,1)$, confirming consistency with Remark 4.

2.5 The policy game

The Stage-1 payoff matrix is

$$\begin{array}{c|cc} & U & D \\ \hline U & \pi_1^{UU},\, \pi_2^{UU} & \bar{\Pi}_1,\, \pi_2^{UD} \\ D & \pi_1^{DU},\, \bar{\Pi}_2 & \pi_1^{DD},\, \pi_2^{DD} \end{array}$$

Table 1 collects the exact closed-form payoffs for all four outcomes under both the Stackelberg standard and the MSNE. Table 2 reports numerical values at selected values of α, and confirms the three-region structure identified in Corollary 1 below.

Table 1: Equilibrium profits for the linear-city model ($k = e\,E_3(1) \approx 0.298$; $E_3(1) = \int_1^\infty e^{-s}s^{-3}\,ds \approx 0.1097$; $\alpha = c/t \in [0, 1)$)

	Stackelberg (TV)		**MSNE (this paper)**	
Outcome	π_1	π_2	π_1	π_2
(U,U)	$\frac{t(3+\alpha)^2}{18}$	$\frac{t(3-\alpha)^2}{18}$	$\frac{t(3+\alpha)^2}{18}$	$\frac{t(3-\alpha)^2}{18}$
(D,D)	$\frac{t(1+\alpha)^2}{4}$	$\frac{t(1-\alpha)^2}{4}$	$\frac{t(1+\alpha)^2}{4}$	$\frac{t(1-\alpha)^2}{4}$
$(U,D)^a$	$\frac{t(1+\alpha)^2}{8}$	$\frac{t(3-\alpha)^2}{16}$	$\frac{t(1+\alpha)^2}{8}$	$\frac{t(1+\alpha)^2 k}{16} + \frac{t(1-\alpha)(3-\alpha)}{8}$
$(D,U)^b$	$\frac{t(3+\alpha)^2}{16}$	$\frac{t(1-\alpha)^2}{8}$	$\frac{t(1-\alpha)^2 k}{16} + \frac{t(1+\alpha)(3+\alpha)}{8}$	$\frac{t(1-\alpha)^2}{8}$

In every asymmetric subgame the mill-pricing firm earns the same payoff under Stackelberg and MSNE; only the discriminating firm's profit falls.

[a] firm 1 (cost 0) mills; firm 2 (cost αt) discriminates.

[b] firm 2 (cost αt) mills; firm 1 (cost 0) discriminates.

Proposition 7 ((D,D) is always Nash)**.** *(D,D) is a Nash equilibrium for all $\alpha \in [0,1)$.*

Proof. Firm 1: $\pi_1^{DD} = t(1+\alpha)^2/4 > t(1+\alpha)^2/8 = \bar{\Pi}_1$. Firm 2: $\pi_2^{DD} = t(1-\alpha)^2/4 \geq t(1-\alpha)^2/8 = \bar{\Pi}_2$ always holds. □

Proposition 8 ((U,U) Nash condition)**.** *(U,U) is a Nash equilibrium iff $\alpha < \alpha^{**}$, where*

$\alpha^{**} \approx 0.587$ *is the unique root in* $[0, 1)$ *of*

$$\frac{(3+\alpha)^2}{18} = \frac{(1-\alpha)^2 k}{16} + \frac{(1+\alpha)(3+\alpha)}{8}. \tag{3}$$

Proof. Firm 2's condition $\pi_2^{UU} \geq \pi_2^{UD}$ holds for all $\alpha \in [0, 1)$: one can verify directly that $t(3-\alpha)^2/18 > t(1+\alpha)^2 k/16 + t(1-\alpha)(3-\alpha)/8$ for all $\alpha \in [0, 1)$, since the difference equals $t(3-\alpha)[(3-\alpha)/18 - (1-\alpha)/8] - t(1+\alpha)^2 k/16$, which is strictly positive at $\alpha = 0$ (equal to $t[1/2 - 3t/8 - k/16] \approx 0.106t > 0$) and remains positive throughout $[0, 1)$ as verified numerically in Table 2. Firm 1's condition $\pi_1^{UU} \geq \pi_1^{DU}$ is equation (3), which has a unique root $\alpha^{**} \approx 0.587$ in $[0, 1)$. □

Table 2: Numerical values for the linear-city model at selected $\alpha = c/t$ (profits multiplied by t; (S) = Stackelberg, (M) = MSNE)

	(U,U)		(D,D)		(U,D): π_2/t		(D,U): π_1/t		
α	π_1/t	π_2/t	π_1/t	π_2/t	(S)	(M)	(S)	(M)	Game
0.0	.500	.500	.250	.250	.563	.394	.563	.394	Coord.
0.1	.534	.467	.303	.203	.526	.349	.601	.441	Coord.
0.2	.569	.436	.360	.160	.490	.307	.640	.492	Coord.
0.3	.605	.405	.423	.123	.456	.268	.681	.545	Coord.
0.4	.642	.376	.490	.090	.423	.232	.723	.602	Coord.
0.5	.681	.347	.563	.063	.391	.198	.766	.661	Coord.
$\alpha^{**}\approx 0.587$	.715	.324	.630	.043	.364	.172	.804	.715*	Coord.
0.6	.720	.320	.640	.040	.360	.168	.810	.723	PD
0.7	.761	.294	.723	.023	.331	.140	.856	.788	PD
$\alpha^{*}\approx 0.784$	.796	.273	.796	.012	.311	.119	.889	.845†	PD
0.8	.802	.269	.810	.010	.303	.115	.903	.856	(D,D)
0.9	.845	.245	.903	.003	.278	.094	.951	.926	(D,D)

Mill-pricer profits $\pi_1^{UD} = t(1+\alpha)^2/8$ and $\pi_2^{DU} = t(1-\alpha)^2/8$ are identical under both conventions and omitted.

* At α^{**}: $\pi_1^{UU} = \pi_1^{DU}$(M). For $\alpha \in [\alpha^{**}, \alpha^{*})$ the game is a Prisoner's Dilemma.

† At $\alpha^{*} = (-3+6\sqrt{2})/7$: $\pi_1^{UU} = \pi_1^{DD}$. Above α^{*}, Firm 1 prefers (D,D) to (U,U), eliminating the Prisoner's Dilemma structure.

Corollary 1 (Three-region classification)**.** *Let* $\alpha^{*} = (-3+6\sqrt{2})/7 \approx 0.784$ *be the positive root of* $7\alpha^2 + 6\alpha - 9 = 0$ *(i.e.* $\pi_1^{UU} = \pi_1^{DD}$*).*

- $\alpha \in [0, \alpha^{**})$*:* Coordination game. *Both* (U, U) *and* (D, D) *are Nash;* (U, U) *Pareto-dominates.*

- $\alpha \in [\alpha^{**}, \alpha^{*})$*:* Prisoner's Dilemma. (D, D) *is the unique Nash equilibrium;* (U, U) *Pareto-dominates for both firms.*

- $\alpha \in [\alpha^{*}, 1)$*:* (D, D) Nash, no conflict. (D, D) *is the unique Nash equilibrium, but firm 1 prefers* (D, D) *to* (U, U)*, so the Prisoner's Dilemma structure disappears.*

2.6 The case $\alpha = 0$: symmetric firms

When $\alpha = c/t = 0$ both firms have the same marginal cost ($c_1 = c_2 = 0$) and are located symmetrically at the two endpoints of the segment. The model reduces to the standard symmetric Hotelling duopoly. All formulas derived above simplify considerably and the two asymmetric subgames (U, D) and (D, U) become identical by symmetry, so a single MSNE characterization covers both.

Symmetric subgames. Under mutual uniform pricing (U, U), both firms charge the standard Hotelling price $p_1^* = p_2^* = t$, each sells to half the market, and each earns $\pi^{UU} = t/2$. Under mutual discrimination (D, D), the market boundary is at $x^{DD} = 1/2$ (the midpoint), each firm prices at the rival's delivered cost at every location, and each earns $\pi^{DD} = t/4$. Hence $\pi^{UU} = 2\pi^{DD}$: mutual uniform pricing doubles each firm's profit relative to mutual discrimination.

The asymmetric subgame under the Stackelberg standard. By symmetry, the (U, D) and (D, U) subgames are identical at $\alpha = 0$. Under TV's Stackelberg standard, the mill-pricing firm (say firm 1) sets $p_1^* = t/2$, earning $\pi_1^{UD} = t/8$. The discriminating firm (firm 2) earns $\pi_2^{UD} = t(3 - 0)^2/16 = 9t/16$. Since $9t/16 > t/2 = \pi^{UU}$, discrimination strictly dominates uniform pricing for firm 2, generating TV's Prisoner's Dilemma.

The asymmetric subgame under simultaneous competition (MSNE). Under the MSNE at $\alpha = 0$, firm 1 randomizes over the support $[p_L, p_H] = [t/4,\, t/2]$ with CDF

$$F_1(p_1) = 1 - \frac{t}{4}\, e^2 \cdot \frac{\exp\bigl(-t/(2(t/2 - p_1))\bigr)}{t/2 - p_1}, \quad p_1 \in [t/4,\, t/2].$$

Firm 1 earns $\bar{\Pi}_1 = t/8$ for every realization of its mill price — the same as under the Stackelberg standard. Firm 2's equilibrium schedule has two parts:

- $x \in [1/2,\, 1]$: $p_2^*(x) = t/4 + tx$ (certain capture: firm 2 wins this interval regardless of firm 1's price);

- $x \in [1/4,\ 1/2]$: $p_2^*(x) = t/(8x) + tx$ (probabilistic capture: firm 2 wins location x only if $p_1 > t/(8x)$).

The discriminating firm's expected profit is

$$\pi_2^{UD}\Big|_{\alpha=0} = \frac{tk}{16} + \frac{3t}{8} = \left[\frac{e\,E_3(1)}{16} + \frac{3}{8}\right] t \approx 0.394\,t,$$

where $k = e\,E_3(1) \approx 0.298$ and $E_3(1) = \int_1^\infty e^{-s}s^{-3}\,ds \approx 0.1097$. The certain part integrates to $3t/8$ (from $x \in [1/2, 1]$) and the probabilistic part contributes $tk/16$. This is the exact value that Aguirre and Martín 2001 showed lies below $\pi^{UU} = t/2$ but did not compute.

The policy game at $\alpha = 0$. The Stage-1 payoff matrix (using MSNE payoffs for the asymmetric subgames) is:

	U	D
U	$t/2,\ t/2$	$t/8,\ 0.394\,t$
D	$0.394\,t,\ t/8$	$t/4,\ t/4$

There are two Nash equilibria:

- (U, U) is Nash because $t/2 > 0.394\,t$ (neither firm wishes to deviate from mutual uniform pricing to discrimination given the other mills uniformly).

- (D, D) is Nash because $t/4 > t/8$ (neither firm wishes to deviate from mutual discrimination to uniform pricing given the other discriminates).

Since $\pi^{UU} = t/2 > t/4 = \pi^{DD}$, mutual uniform pricing Pareto dominates mutual discrimination. The game is a *coordination game*: TV's Prisoner's Dilemma completely disappears once the Stackelberg standard is replaced by simultaneous competition.

The key mechanism is transparent at $\alpha = 0$. Under the Stackelberg standard, firm 2 earns $9t/16 \approx 0.563\,t$, which exceeds $\pi^{UU} = t/2$. Under simultaneous competition, firm 1's randomization prevents firm 2 from exploiting a known price, compressing its expected profit to $0.394\,t < t/2$. This fall of approximately $0.169\,t$ is sufficient to overturn the incentive to discriminate.

3 The Product Differentiation Model

3.1 Model

Consumers are distributed uniformly over a unit circle $[0, 1]$, where the location of each consumer can be interpreted as its preferred product variety. Each consumer buys a unit of the product from the firm with the lowest final price. Two firms are located at opposite points: firm 1 at $x = 0$ and firm 2 at $x = 1/2$.[5] Transport costs are quadratic: the transport cost from a firm located at a to a consumer located at x is $t(d(x,a))^2$, where $d(x,a) = \min(|x-a|, 1-|x-a|)$ is the arc distance and $t > 0$. Production is costless. There is a two-stage game: in Stage 1, each firm independently commits to either uniform pricing (U) or discriminatory pricing (D); in Stage 2, firms compete on price. I solve by backward induction.

3.2 Symmetric subgames

Lemma 5 (Uniform equilibrium)**.** *Under (U,U): $p^* = t/4$ and $\Pi^{UU} = t/8$.*

Proof. On the left arc, the indifferent consumer satisfies $p_1 + tx^2 = p_2 + t(1/2 - x)^2$, giving $\hat{x}_L = 1/4 + (p_2 - p_1)/t$. Symmetry yields on the right arc $\hat{x}_R = 3/4 + (p_1 - p_2)/t$. Residual demands of firm 1 and firm 2 are, respectively, $D_1(p_1,p_2) = 1/2 + 2(p_2 - p_1)/t$ and $D_2(p_1,p_2) = 1/2 + 2(p_1 - p_2)/t$. From the first-order conditions of the profit maximization problems, it is direct to obtain $p^* = t/4$, with each firm selling $1/2$ and earning $\Pi^{UU} = t/8$. □

Lemma 6 (Discriminatory equilibrium)**.** *Under (D,D): $p^*(x) = \max\{t(d(x,0))^2,\, t(d(x,1/2))^2\}$ and $\Pi^{DD} = t/16$.*

Proof. Following Lederer and Hurter (1986) and TV, at each location the equilibrium price is the maximum of the transport costs. That is,

$$p^*(x) = \max\bigl\{t(d(x,0))^2,\, t(d(x,\tfrac{1}{2}))^2\bigr\} = \begin{cases} \max\{tx^2,\, t(\frac{1}{2}-x)^2\} & 0 \le x \le \frac{1}{2}, \\ \max\{t(1-x)^2,\, t(x-\frac{1}{2})^2\} & \frac{1}{2} \le x \le 1. \end{cases}$$

The market boundaries are $1/4$ and $3/4$. Equilibrium profits are

$$\Pi_1^{DD} = \Pi^{DD} = \int_0^{1/4} \bigl[p^*(x) - tx^2\bigr]\,dx + \int_{3/4}^{1} \bigl[p^*(x) - t(1-x)^2\bigr]\,dx = \frac{t}{16},$$
$$\Pi_2^{DD} = \Pi^{DD} = \int_{1/4}^{1/2} \bigl[p^*(x) - t(\tfrac{1}{2}-x)^2\bigr]\,dx + \int_{1/2}^{3/4} \bigl[p^*(x) - t(x-\tfrac{1}{2})^2\bigr]\,dx = \frac{t}{16}. \qquad \square$$

□

Remark 8. $\Pi^{UU} = 2\Pi^{DD}$.

[5] Alternatively, firms are located at l_1 and l_2 and the distance between them is $1/2$.

3.3 The asymmetric (U, D) subgame

Firm 1 mills at price $p_1 \geq 0$; firm 2 sets delivered prices $p_2(x)$. Consumer x buys from firm 1 iff $p_1 + t\,d(x, 0)^2 < p_2(x)$. No pure-strategy Nash equilibrium exists in this subgame under simultaneous price competition.

Solution under the Stackelberg standard. Under TV's Stackelberg standard firm 1 sets $p_1^* = t/8$, earning $\bar{\Pi} = t/32$, while firm 2 earns $9t/64 > t/8 = \Pi^{UU}$, making discrimination dominant.

Solution under simultaneous competition (MSNE). Under simultaneous competition firm 1 randomizes. I derive the unique MSNE following four steps: support, expected profit, first-order condition, and CDF.

Step 1: The support $[p_L, p_H]$.

Equilibrium schedule. On the left arc the indifference condition $p_1\tilde{x} = \bar{\pi}_L$ (constant) gives $p_2^*(x) = \bar{\pi}_L/x + tx^2$. Feasibility $p_2^*(x) \geq t(1/2 - x)^2$ requires $\bar{\pi}_L \geq tx(1/4 - x)$.

Lemma 7 (Feasibility floor, (U, D) circular)**.** *The constraint $p_2^*(x) \geq t(d(x, 1/2))^2$ gives $\bar{\pi}_L = t/64$, binding at $x^* = 1/8$.*

Proof. $tx(1/4 - x)$ is maximized at $x^* = 1/8$ with value $t/64$. Hence $\bar{\pi}_L \geq t/64$; firm 2 minimizes $\bar{\pi}_L$ subject to feasibility, so $\bar{\pi}_L = t/64$. Symmetry gives $\bar{\pi}_R = t/64$. □

Firm 1's MSNE payoff is $\bar{\Pi} = 2\bar{\pi}_L = t/32$, equal to the Stackelberg payoff.

Upper boundary p_H. For firm 1 to be indifferent at the top of its support, firm 2's profit at the boundary location must equal zero. Firm 2's profit at boundary location $x = \bar{\pi}_L/p_1$ equals $(p_1 - t/8)^2/p_1$, which vanishes at $p_H = t/8$. We verify below (Step 4) that $F_1(p_H) = 1$.

Lower boundary p_L. At the lowest price p_L in the support, firm 1's share on the left arc reaches the natural (D, D) boundary $x^{DD} = 1/4$, i.e. $\tilde{x}(p_L) = \bar{\pi}_L/p_L = 1/4$. Solving:

$$p_L = \frac{\bar{\pi}_L}{1/4} = \frac{t/64}{1/4} = \frac{t}{16}.$$

Step 2: firm 2's expected profit. Given F_1, the CDF of firm 1's mill price, at each location x firm 2 sets $p_2(x)$ to maximize its expected profit. The expected profit at x on the left arc is

$$\pi_2(x) = \left[p_2(x) - t(d(x, \tfrac{1}{2}))^2\right] \cdot \left[1 - F_1(R(x))\right],$$

where $R(x) \equiv p_2(x) - tx^2$ is the reserve price.

Step 3: First-order condition. The first-order condition for firm 2 at each location yields the ODE

$$\frac{d\ln(1-F_1)}{dp_1} = -\frac{p_1}{(p_1 - t/8)^2}. \tag{4}$$

The denominator is a perfect square because $\bar{\pi}_L = t/64$ makes the discriminant of $p_1^2 - (t/4)p_1 + t^2/64$ equal to zero.

Step 4: CDF. Integrating the ODE gives the unique CDF stated in the following proposition.

Proposition 9 (Unique MSNE, circular model)**.** *The unique mixing distribution for firm 1 has support* $[t/16,\, t/8]$ *with CDF*

$$F_1(p_1) = 1 - \frac{t}{16}\, e^2 \cdot \frac{\exp\bigl(-t/\bigl(8(t/8 - p_1)\bigr)\bigr)}{t/8 - p_1}, \quad p_1 \in [t/16,\, t/8],$$

with exponent $t/(8\varepsilon_L) = 2$ *at* $p_L = t/16$*. The equilibrium schedule has two parts:*

- $x \in [1/8, 1/4]$*:* $p_2^*(x) = t/(64x) + tx^2$ *(probabilistic capture);*
- $x \in (1/4, 1/2]$*:* $p_2^*(x) = t/16 + tx^2$ *(certain capture).*

By symmetry the right-arc schedule is analogous. Given F_1*, firm 2's unique best response is the equilibrium schedule, so the MSNE is unique.*

The value $p_L = t/16$ is confirmed by requiring F_1 to be a proper CDF with total mass 1: the substitution $u = t/(8\varepsilon)$ in the density f_1 gives lower limit $u_L = t/(8\varepsilon_L) = 2$, which holds exactly when $\varepsilon_L = t/16$, i.e. $p_L = p_H - \varepsilon_L = t/8 - t/16 = t/16$.

Verification. As $p_1 \to p_H$, i.e. $\varepsilon \to 0$, $1 - F_1 = K\varepsilon^{-1}\exp(-t/(8\varepsilon)) \to 0$, so $F_1(p_H) = 1$, confirming $p_H = t/8$ is the upper boundary.

Proposition 10 (firm 2's expected payoff, Section IIIB)**.**

$$\Pi_2^{UD} = \frac{t\, e\, E_3(1)}{64} + \frac{3t}{32} \approx 0.098\, t.$$

Proof. By two-arc symmetry, $\Pi_2^{UD} = 2\,E[\Pi_2^{\text{left}}]$.

Certain part ($x \in [1/4, 1/2]$): profit $t/16 + t(x - 1/4)$ integrates to $3t/64$.

Probabilistic part ($x \in [1/8, 1/4]$): substituting $p = t/(64x)$ gives profit $(p - t/8)^2/p$. Then $u = t/(8\varepsilon)$ with lower limit $u_L = 2$, followed by $w = u - 1$, yields $t\, e\, E_3(1)/128$ via $\int_1^\infty e^{-w} w^{-3}\, dw = E_3(1)$. Doubling and summing gives the result. □

Remark 9 (Symmetry of the asymmetric subgames, Section IIIB). Because production is costless and the two firms are symmetrically located on the circle, the subgames (U, D) and (D, U) are identical. Whichever firm sets the uniform mill price plays the role of firm 1 above, and the payoffs $\bar{\Pi} = t/32$ (mill pricer) and $\Pi_2^{UD} \approx 0.098\,t$ (discriminator) apply equally to both cases. In particular, the policy-game matrix is symmetric, and a single MSNE characterization covers both asymmetric subgames.

3.4 The policy game

By symmetry the Stage-1 payoff matrix is

$$\begin{array}{c|cc} & U & D \\ \hline U & \Pi^{UU},\, \Pi^{UU} & \bar{\Pi},\, \Pi_2^{UD} \\ D & \Pi_2^{UD},\, \bar{\Pi} & \Pi^{DD},\, \Pi^{DD} \end{array}$$

The exact payoffs for both conventions are collected in Table 3.

Table 3: Equilibrium profits for the circular market model ($k = e\,E_3(1) \approx 0.298$; $E_3(1) = \int_1^\infty e^{-s}s^{-3}\,ds \approx 0.1097$; $c = 0$; profits in units of t)

	Stackelberg (TV)		**MSNE (this paper)**	
Outcome	Π_1	Π_2	Π_1	Π_2
(U, U)	$t/8$	$t/8$	$t/8$	$t/8$
(D, D)	$t/16$	$t/16$	$t/16$	$t/16$
$(U, D)^c$	$t/32$	$9t/64$	$t/32$	$\left[\frac{e\,E_3(1)}{64} + \frac{3}{32}\right] t$

In the asymmetric subgame the mill-pricing firm earns the same payoff under Stackelberg and MSNE; only the discriminating firm's profit falls.

[c] Symmetric subgame ($c = 0$): (U, D) and (D, U) coincide; firm 1 mills and firm 2 discriminates.

Corollary 2 (Coordination game, Section IIIB)**.** *Both (D, D) and (U, U) are Nash equilibria. The policy game is a coordination game: (U, U) Pareto-dominates (D, D) since $\Pi^{UU} = t/8 > t/16 = \Pi^{DD}$. TV's Prisoner's Dilemma in Section IIIB does not hold under simultaneous price competition.*

Proof. (D, D) Nash: $\Pi^{DD} = t/16 > t/32 = \bar{\Pi}$. (U, U) Nash: $\Pi^{UU} = t/8 > \Pi_2^{UD} \approx 0.098\,t$. □

4 Discussion

The Gabszewicz–Thisse observation. Gabszewicz and Thisse (1992) remark that price discrimination enlarges the strategy space relative to uniform pricing in the same way that mixed strategies enlarge it relative to pure strategies. My analysis gives this observation precise content. In every MSNE of every asymmetric subgame, the two firms play qualitatively different types of strategy: the mill-pricing firm randomizes over its mill price while the discriminating firm plays a *pure* delivered-price schedule. This asymmetry is not incidental: the discriminating firm's schedule is a deterministic best response to the mill pricer's distribution, and it is precisely because the mill pricer randomizes that the discriminating firm can no longer exploit a known price. Under the Stackelberg convention the mill pricer commits to a single price, and the discriminating follower can position its schedule just below the rival's delivered price at every location, reaping an exceptionally high payoff. Under simultaneous competition the random mill price forces the discriminator to set a defensive schedule without knowing the rival's price, substantially compressing its expected revenue. The interplay between the uniform pricer's mixed strategy and the discriminator's pure strategy is therefore the central mechanism driving all our results.

Why the Prisoner's Dilemma disappears. In both models the discriminator's payoff falls strictly when the Stackelberg convention is replaced by the MSNE. For small enough cost asymmetry (i.e. α small), this fall is sufficient to bring the discriminator's payoff below the uniform-pricing payoff, so that neither firm wishes to deviate from mutual uniform pricing. This restores (U, U) as a Nash equilibrium alongside (D, D), turning the Prisoner's Dilemma into a coordination game. In the circular model the result is unconditional: because costs are symmetric the fall in the discriminator's payoff always suffices, and the game is a coordination game for all parameter values.

Mill-pricer invariance. A striking feature of both models is that the mill-pricing firm's MSNE payoff equals its Stackelberg payoff exactly. This is not a coincidence: the mill pricer's equilibrium profit equals the feasibility floor $\bar{\pi}$ (or $\bar{\pi}_2$), which is determined by the requirement that the discriminatory firm's schedule be feasible everywhere, independently of how the mill pricer mixes. For any mixing distribution F, the mill pricer earns $\bar{\pi}$ for every realization of its price; the mixing distribution affects only the discriminating firm's expected revenue. Hence the entire difference between the Stackelberg and MSNE conventions is borne by the discriminating firm alone.[6]

Relation to Aguirre and Martín 2001. Aguirre and Martín (2001) work within TV's Section IIIA model at $\alpha = 0$. They show by an inequality that (U, U) is a Nash equilibrium, overturning TV's Prisoner's Dilemma, but do not derive the exact discriminating-firm payoff. My formula at $\alpha = 0$ supplies the exact figure: both asymmetric subgames coincide by symmetry and give the discriminating firm the payoff $[e\, E_3(1)/16 + 3/8]\, t \approx 0.394\, t < \pi^{UU} = t/2$, confirming their conclusion with a precise value. The present paper extends their result to all $\alpha \in [0, 1)$, adds the circular model, and supplies exact payoffs and game-type classification throughout.

5 Conclusion

I have derived the unique mixed-strategy Nash equilibrium for every asymmetric subgame in both models of TV, obtaining exact closed-form payoffs throughout.

In the circular product differentiation model (Section IIIB) the policy game is always a coordination game. The discriminating firm's payoff under the MSNE falls strictly below the uniform-pricing payoff, so mutual uniform pricing and mutual discrimination are both Nash equilibria, with the former Pareto-dominant. TV's Prisoner's Dilemma in this model does not survive simultaneous price competition.

In the linear spatial competition model (Section IIIA) with asymmetric marginal costs the game has three regions depending on the degree of cost asymmetry. For sufficiently small cost differences the game is a coordination game; for intermediate cost differences it is a genuine Prisoner's Dilemma, where mutual discrimination is the unique Nash equilibrium yet both firms would prefer mutual uniform pricing; and for large cost differences mutual discrimination is again the unique Nash equilibrium but without Prisoner's Dilemma tensions, since the low-cost firm actually prefers discriminating. In the equal-cost case the exact discriminating-firm payoff obtained here confirms and quantifies the result of Aguirre and Martín 2001.

It is worth noting that several papers in the literature have adopted TV's Stackelberg

[6] Both models also share a common algebraic structure. In each asymmetric subgame the net-revenue function takes the form of a perfect square divided by the mill price, $(p - p_H)^2/[\text{cost factor}]$, because the feasibility floor drives the discriminant of the net-revenue quadratic to zero, creating a double root at p_H. As a consequence, all three ODEs (1), (2), and (4) integrate to the same functional form $G(p) = A \exp(-B/\varepsilon)/\varepsilon$ with exponent 2 at the lower support boundary, and the discriminating firm's exact payoff in every case is expressed via the single special function $E_3(1)$.

standard when analyzing asymmetric subgames in related models (Liu and Serfes, 2004; Shaffer and Zhang, 2002; Ghose and Huang, 2009; Matsumura and Matsushima, 2015). Eber (1997), for instance, analyzes the strategic choice of price policy while also endogenizing firms' locations in the market, following TV's Stackelberg standard for the asymmetric subgame. Aguirre (2026) derives the MSNE in a model of spatial price discrimination in a linear city with quadratic transport costs, allowing for endogenous location choice. Replacing the Stackelberg standard with the MSNE changes the results—sometimes drastically: equilibria that appeared as Prisoner's Dilemmas become coordination games, and the incentives for price discrimination are substantially weakened.

Running through both models, two structural results hold in every asymmetric subgame. First, mill-pricer invariance: the mill-pricing firm earns the same payoff whether competition is Stackelberg or simultaneous, while the entire difference between the two standards falls on the discriminating firm. Second, the interplay between the mill pricer's mixed strategy and the discriminator's pure strategy is the central mechanism: randomization by the mill pricer prevents the discriminating firm from exploiting a known price, compressing its payoff and, for small enough cost asymmetry, restoring the incentive to price uniformly.

A final remark concerns the modeling choice itself. Some authors have adopted the Stackelberg standard on the grounds that deriving the MSNE of the asymmetric subgame is prohibitively complex. The present paper shows, building on the insight of Gabszewicz and Thisse (1992) that price discrimination enlarges the strategy space in the same way that mixed strategies do, that the resolution is not intractable: the observation that the mill-pricing firm randomizes while the discriminating firm plays a pure schedule makes the structure of the MSNE transparent, and closed-form payoffs are obtainable.

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