Ecuación lineal de 1er orden

$$\begin{cases} a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = d(x,y) \\ u(f(s), g(s)) = h(s) \end{cases} \qquad (0)$$

$a, b, c \in C^1(U)$, $d \in C(U)$, $P = (f(s_0), g(s_0)) \in U$

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* Buscamos una superficie $z = u(x,y)$ de clase C^1, que cerca de $(f(s_0), g(s_0), h(s_0))$ pase por la curva

$$\{(f(s), g(s), h(s)) : s \in I\}, \quad s_0 \in I, \quad I \text{ intervalo}$$

y cuyo vector normal en $(x, y, u(x,y))$

$$N = \frac{(u_x, u_y, -1)}{\sqrt{1 + u_x^2 + u_y^2}} \quad \text{en} \quad X = (x, y, z) = (x, y, u(x,y))$$

verifique

$$N_X \cdot (a(x,y), b(x,y), -c(x,y).z + d(x,y)) = 0$$

Si $X = (x(s,t), y(s,t), z(s,t))$ es una parametrización de dicha superficie, se debería de verificar que

- $z(s,t) = u(x(s,t), y(s,t))$
- $x(s,0) = f(s)$, $y(s,0) = g(s)$, $z(s,0) = h(s)$
- $$N_X = \frac{X_s \wedge X_t}{|X_s \wedge X_t|} = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix}$$

y necesariamente

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \neq 0, \quad \text{para } (s,t) \text{ cerca de } (s_0, 0).$$

• Además,

$$(a(x,y),\ b(x,y),\ -c(x,y)z + d(x,y))$$

será tangente a la superficie y podría ser factible elegir o encontrar la parametrización de forma que

$$\left\{\begin{aligned} \frac{\partial x}{\partial t} &= a(x,y), & x(s,0) &= f(s) \\ \frac{\partial y}{\partial t} &= b(x,y), & y(s,0) &= g(s). \\ \frac{\partial z}{\partial t} &= -c(x,y)z + d(x,y). \end{aligned}\right. \quad (1)$$

En general, si $(x(s,t),\ y(s,t))$ son soluciones de (1) y $z = u(x,y)$ es solución cerca de I

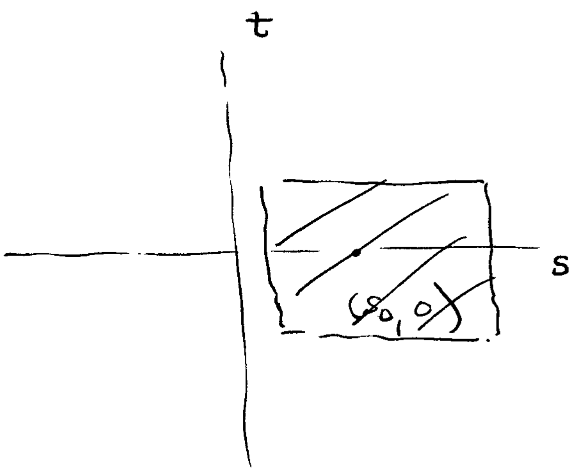
$$\frac{\partial}{\partial t} z(s,t) = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t}$$

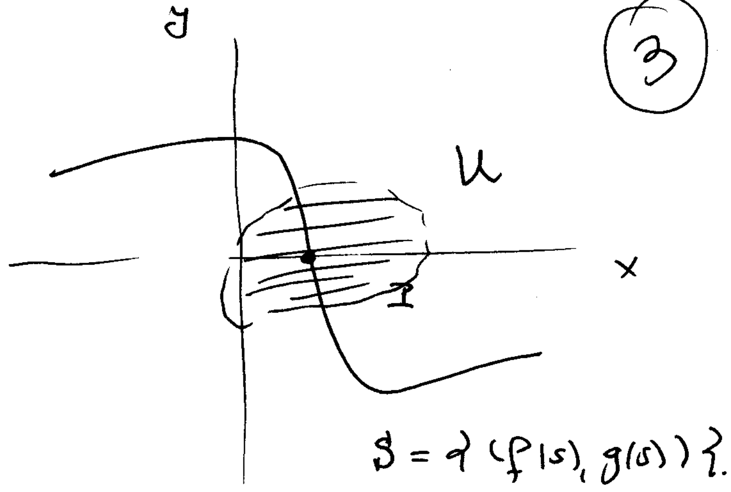
$$= a u_x + b u_y = d - cz.$$

Las curvas solución de

$$\left\{\begin{aligned} \frac{\partial x}{\partial t} &= a(x,y), & x(s,0) &= f(s), \\ \frac{\partial y}{\partial t} &= b(x,y), & y(s,0) &= g(s), \\ \frac{\partial z}{\partial t} &= d(x,y) - c(x,y)\, z(x,y), & z(s,0) &= h(s). \end{aligned}\right.$$

se dicen curvas características de (0).





Si $S = \{(f(s), g(s)) : s \in I\}$ es no característica en P,

$$\begin{vmatrix} f'(s) & g'(s) \\ a(f(s), g(s)) & b(f(s), g(s)) \end{vmatrix} \neq 0, \quad \text{en} \quad s = s_0.$$

Por TEEDO, TDP y TFI, $\exists\, \delta > 0$ t.q.

$$\begin{aligned} (s_0 - \delta, s_0 + \delta) \times (-\delta, \delta) &\longleftrightarrow \mathcal{U} \\ (s, t) &\longmapsto (x(s,t), y(s,t)) \end{aligned}$$

es difeomorfismo C^1 y

$$\begin{cases} x = x(s,t) \\ y = y(s,t) \end{cases} \text{tiene inversa local} \quad \begin{cases} s = s(x,y) \\ t = t(x,y) \end{cases}$$

entre sendos entornos de $(s_0, 0)$ y P. Definimos

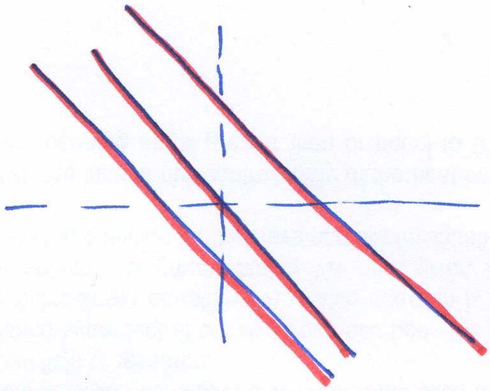
$$u(x,y) = z(s,t) = z(s(x,y), t(x,y)),$$

para $(x,y) \in \mathcal{U}$. Entonces $u \in C^1(\mathcal{U})$ y es solución del problema de Cauchy (O). La solución es única en un entorno de P.

$$\begin{cases} u_x - u_y = 0 \\ u(x,0) = \phi(x) \end{cases} , \quad u = \phi(x+y)$$

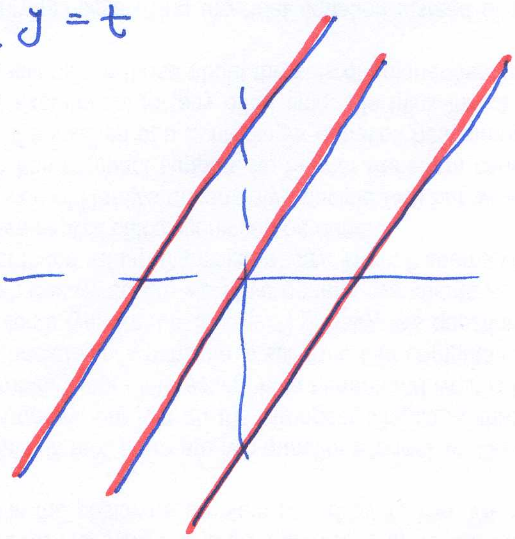
$$\begin{cases} \dfrac{\partial x}{\partial t} = 1, \\ \dfrac{\partial y}{\partial t} = -1, \end{cases} \quad \begin{cases} x = t+s \\ y = -t \end{cases}$$

$$x + y = \text{const.}$$



Ecuación del transporte

$$\begin{cases} u_y + b u_x = 0 \\ u(x,0) = \phi(x) \end{cases} \quad \begin{cases} \dfrac{\partial x}{\partial t} = 1 \\ \dfrac{\partial y}{\partial t} = b \end{cases} , \quad \begin{cases} x = s + bt \\ y = t \end{cases} \quad x - by = \text{const.}$$



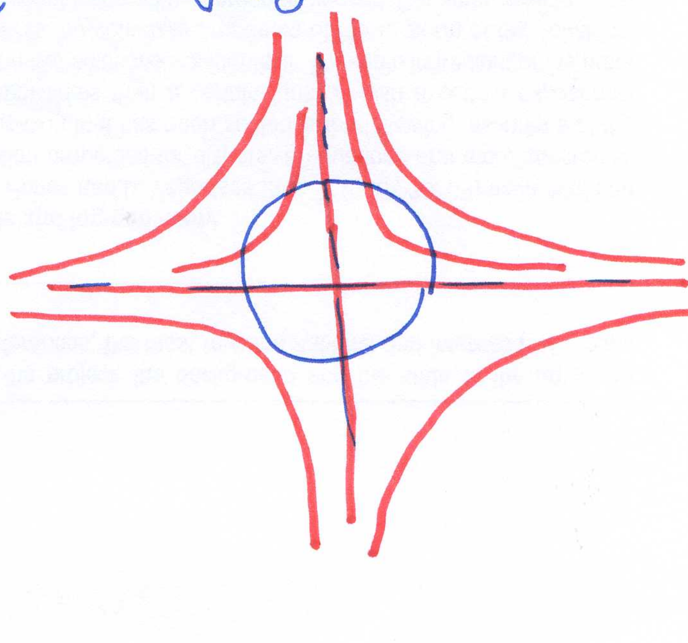
$$u(x,y) = \phi(x - by)$$

Si $\phi(x) = e^{-x^2}$, $u = e^{-(x-by)^2}$.

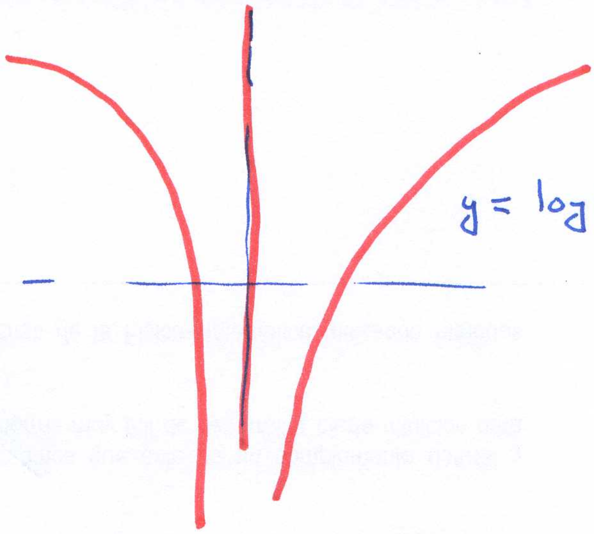
"Onda solitaria"

$$\begin{cases} x u_x - y u_y = 0 \\ u(x,x) = \phi(x) \end{cases} \quad \begin{cases} \dfrac{\partial x}{\partial t} = x \\ \dfrac{\partial y}{\partial t} = -y \end{cases} , \quad \begin{cases} x = s e^{t} \\ y = s e^{-t} \end{cases} , \quad x \cdot y = \text{const.}$$

$$u(x,y) = \phi(\sqrt{xy}) , \quad xy = s^2, \quad t = \log\left(\sqrt{\tfrac{x}{y}}\right)$$



$$\begin{cases} x u_x + u_y = 0 \\ u(x,0) = \phi(x) \end{cases}$$



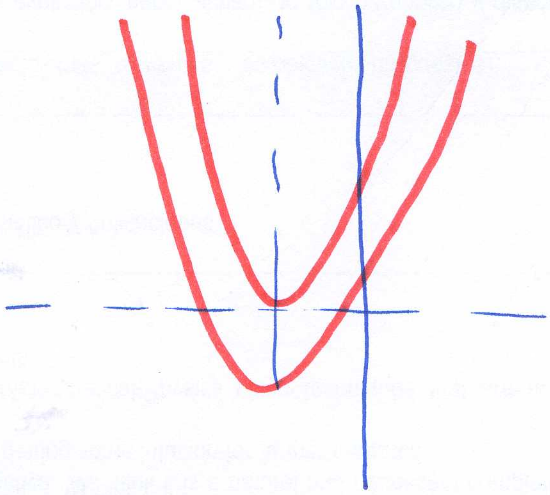
$y = \log\left|\frac{x}{s}\right|$

$$\begin{cases} \frac{\partial x}{\partial t} = x, & x(s,0) = s \\ \frac{\partial y}{\partial t} = 1, & y(s,0) = 0 \\ \frac{\partial z}{\partial t} = 0, & z(s,0) = \phi(s) \end{cases}$$

$x = s e^t, \quad y = t, \quad z = \phi(s)$

$$u = \phi(s) = \phi(x e^{-y}).$$

$$\begin{cases} u_x + x u_y = u \\ u(1,y) = \phi(y). \end{cases}$$



$$\begin{cases} \frac{\partial x}{\partial t} = 1, & x(s,0) = 1 \\ \frac{\partial y}{\partial t} = 1, & y(s,0) = s \\ \frac{\partial z}{\partial t} = z, & z(s,0) = \phi(s) \end{cases}$$

$$\begin{cases} x = t+1 \\ y = \frac{(t+1)^2}{2} + s - \frac{1}{2}, \quad y = \frac{x^2}{2} + s - \frac{1}{2} \\ z = \phi(s) e^t \end{cases}$$

S

$$u(x,y) = \phi\left(y - \frac{x^2}{2} + \frac{1}{2}\right) e^{x-1}$$

Ecuación cuasilineal

$$\begin{cases} a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u). & (2) \\ u(f(s),\, g(s)) = h(s). \end{cases}$$

$S = \{ (f(s), g(s), h(s)) \}$ no característica en s_0 si

$$\begin{vmatrix} a(f(s), g(s), h(s)) & b(f(s), g(s), h(s)) \\ f'(s) & g'(s) \end{vmatrix} \neq 0, \text{ a } s = s_0.$$

Motivación: $z = u(x,y)$ es superficie tal que

$$(u_x, u_y, -1) \perp (a(x,y,u), b(x,y,u), c(x,y,u)).$$

$$\begin{cases} \dfrac{\partial x}{\partial t} = a(x,y,z), & x(s,0) = f(s) \\ \dfrac{\partial y}{\partial t} = b(x,y,z), & y(s,0) = g(s) \\ \dfrac{\partial z}{\partial t} = c(x,y,z), & z(s,0) = h(s) \end{cases}$$

$(x(s,t), y(s,t), z(s,t))$ una parametrización de $z = u(x,y)$ cerca de $P = (f(s_0), g(s_0), h(s_0))$.

$$\begin{cases} x = x(s,t) \\ y = y(s,t) \end{cases}, \quad \overline{X} = (x(s,t)\ y(s,t))$$

$$\det J\overline{X}(0,s) = \begin{vmatrix} a(f(s), g(s), h(s)) & b(f(s), g(s), h(s)) \\ f'(s) & g'(s) \end{vmatrix} \neq 0$$

si $s = s_0$ y $\overline{X} : (s_0 - \delta, s_0 + \delta) \times (-\delta, \delta) \longleftrightarrow U \subset \mathbb{R}^2$ es difeomorfismo. Definir

$$u(x,y) = z(s(x,y),\, t(x,y)) \quad \text{en } U$$

y u es solución de clase C^1 cerca de P, si a, b y c son C^1 cerca de P.

Si $S = \{ (f(s), g(s), h(s)) \}$ es no característica a P, la solución C^1 de (2) es única en un entorno de P.

Dem.

$$\begin{cases} a(x,y,u^1)u^1_x + b(x,y,u^1)u^1_y = c(x,y,u^1) \\ a(x,y,u^2)u^2_x + b(x,y,u^2)u^2_y = c(x,y,u^2). \\ u^i(f(s),g(s)) = h(s), \quad i=1,2. \end{cases}$$

Sea $W(x,y) = (u^1-u^2)(x,y)$.

$$\begin{aligned} &a(x,y,u^1)W_x + b(x,y,u^1)W_y \\ &= c(x,y,u^1) - a(x,y,u^1)u^2_x - b(x,y,u^1)u^2_y \\ &= [c(x,y,u^1) - c(x,y,u^2)] + [a(x,y,u^2) - a(x,y,u^1)]u^2_x \\ &\quad + [b(x,y,u^2) - b(x,y,u^1)]u^2_y . \end{aligned}$$

$$\begin{cases} \bar{a}(x,y)W_x + \bar{b}(x,y)W_y = d(x,y)W, \\ W(f(s),g(s)) = 0, \end{cases} \qquad \begin{aligned} \bar{a}(x,y) &= a(x,y,u^1), \\ \bar{b}(x,y) &= b(x,y,u^2). \end{aligned}$$

cerca de P. Resolvemos

$$\begin{cases} \dfrac{\partial x}{\partial t} = \bar{a}(x,y), & x(0,s) = f(s) \\ \dfrac{\partial y}{\partial t} = \bar{b}(x,y), & y(0,s) = g(s) \end{cases}, \qquad X = (x(s,t), y(s,t))$$

$$\det J\, X(0,s) = \begin{vmatrix} a(f(s),g(s),h(s)) & b(f(s),g(s),h(s)) \\ f'(s) & g'(s) \end{vmatrix} \neq 0,$$

cerca de $s = s_0$.

$$X : (s_0-\delta, s_0+\delta) \times (-\delta,\delta) \longleftrightarrow U \subset \mathbb{R}^2,$$

U abierto alrededor de P es C^1-difeomorfismo. y si

$$z(t,s) = W(x(s,t), y(s,t))$$

$$\begin{cases} \dfrac{\partial z}{\partial t} = d(x,y)z \\ z(s,t) = 0 \end{cases}, \qquad \therefore \quad z(s,t) = 0, \text{ si } |t| \le \delta,$$

$|s-s_0| \le \delta$. Es decir, $u^1 = u^2$ en U.