TDP

Tma: $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, $F \in C^{(k)}(\Omega)$, $k \geq 1$, $x = x(\xi, t)$, denota la solución de

$$\begin{cases} \dfrac{\partial x}{\partial t} = F(x), \\ x(\xi, 0) = \xi, \quad \xi \in \Omega. \end{cases}$$

Entonces, fijado $\xi_0 \in \Omega$, existe $\delta > 0$ tal que la función

$$B_\delta(\xi_0) \times (-\delta, \delta) \subset \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$(\xi, t) \longmapsto x(\xi, t)$$

está bien definida, es de clase $C^{(k)}$ en (ξ, t) y de clase $C^{(k+1)}$ en t. Además, la matriz

$$U(\xi, t) = \left(\frac{\partial x^i}{\partial \xi_j}(\xi, t) \right)_{i,j=1}^{n},$$

es la solución de

$$\begin{cases} \dfrac{\partial U}{\partial t} = J(\xi, t) . U, \\ U(0) = I, \end{cases}$$

donde I es la matriz identidad $n \times n$ y

$$J(\xi, t) = \left(\frac{\partial F^i}{\partial x_j}(x(\xi, t)) \right)_{i,j=1}^{n}.$$

Coordenadas adecuadas.

$P \in \mathbb{R}^n$, Φ función C^k cerca de $P = (p', p_n)$, $\nabla\Phi(P) \neq 0$,

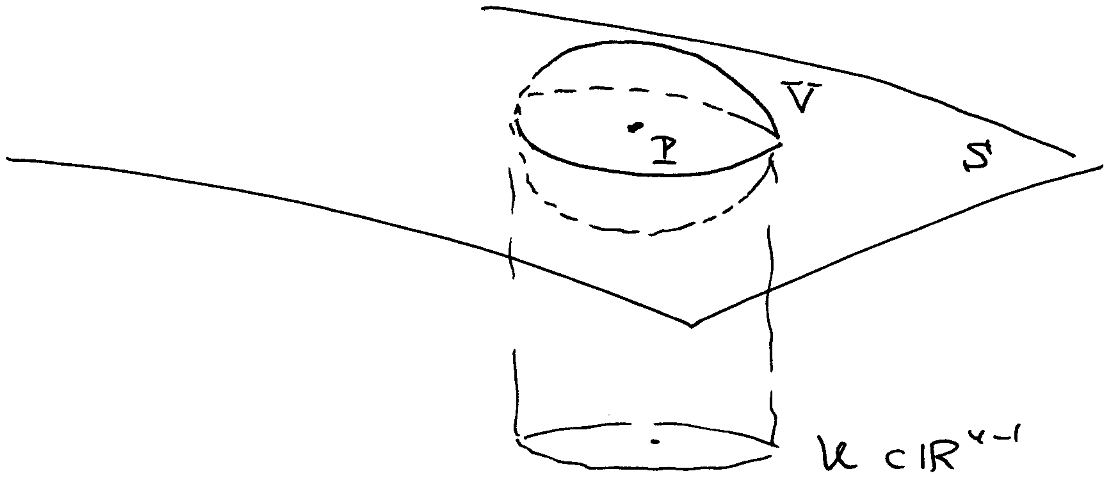
podemos suponer $\frac{\partial \Phi}{\partial x_n}(P) \neq 0$. $S = \{x : \Phi(x) = 0\}$, $x = (x_1, x_2, \dots, x_n)$

Por TFI, $\exists\, \mathcal{U}, V$ abiertos alrededor de $p' \in \mathbb{R}^{n-1}$ y P respectivamente y $\exists\, \varphi \in C^{k-1}(\mathcal{U})$, $\varphi(p') = p_n$,

$$\varphi : \mathcal{U} \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$$

tal que

$$S \cap V = \{ (y, \varphi(y)) : y \in \mathcal{U} \}, \quad y = (y_1, y_2, \dots, y_{n-1})$$



Si $x = (y, \varphi(y))$, $n_x = \dfrac{(\nabla_y \varphi(y), -1)}{\sqrt{1 + |\nabla\varphi(y)|^2}}$, $d\sigma = \sqrt{1 + |\nabla\varphi(y)|^2}\, dy$

¿ Es posible escribir x cerca de P de forma única como

$$x = Q + t\, n_Q, \quad Q \in S \text{ cerca de } P, \quad |t| \leq \delta$$

para algún $\delta > 0$?

¿ Es el sistema de ecuaciones

$$\begin{cases} x_1 = y_1 + t \dfrac{\partial_1 \varphi(y)}{\sqrt{1 + |\nabla\varphi(y)|^2}}, \\ \quad \vdots \\ x_{n-1} = y_{n-1} + t \dfrac{\partial_{n-1} \varphi(y)}{\sqrt{1 + |\nabla\varphi(y)|^2}}, \\ x_n = \varphi(y) - t / \sqrt{1 + |\nabla\varphi(y)|^2}, \end{cases}$$

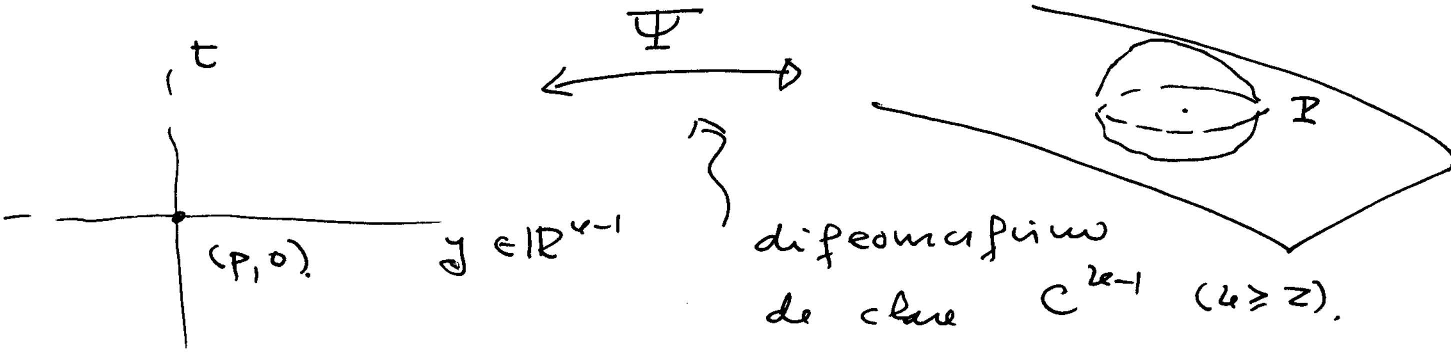
invertible entre abiertos alrededor de P y $(p, 0)$?

$$(x_1, x_2, \dots, x_n) = \Psi(y_1, \dots, y_{n-1}, t).$$

$$D_{(y,0)}\Psi = \begin{pmatrix} 1 & 0 & \cdots & 0 & \partial_1\varphi/\Delta \\ 0 & 1 & \cdots & 0 & \partial_2\varphi/\Delta \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & \partial_{n-1}\varphi/\Delta \\ \partial_1\varphi & \partial_2\varphi & \cdots & \partial_{n-1}\varphi & -1/\Delta \end{pmatrix} = \begin{pmatrix} \nabla x_1 \\ \nabla x_2 \\ \vdots \\ \nabla x_{n-1} \\ \nabla x_n \end{pmatrix}$$

$$\det J\Psi(y,0) = -\sqrt{1+|\nabla\varphi(y)|^2},$$

Lo anterior funciona por TFI mientras $k \geq 2$.



$u = u(x)$ función definida alrededor de P, $x \in S$ cerca de P,

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_i} n_i = \frac{1}{\sqrt{1+|\nabla\varphi(y)|^2}} \left[\sum_{i=1}^{n-1} \frac{\partial u}{\partial x_i}\partial_i\varphi - \partial_n u \right](y, \varphi(y)).$$

$$\frac{\partial^2 u}{\partial n^2} = \frac{1}{(1+|\nabla\varphi(y)|^2)} \left[\sum_{i,j=1}^{n-1} \frac{\partial^2 u}{\partial x_i \partial x_j}\partial_i\varphi\partial_j\varphi - 2\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i \partial x_n}\partial_i\varphi + \frac{\partial^2}{\partial x_n^2}u \right]$$

Ahora,

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial t} \quad y \quad \left(\frac{\partial x_1}{\partial t}, \dots, \frac{\partial x_{n-1}}{\partial t}, \frac{\partial x_n}{\partial t}\right) = \frac{(\nabla_y\varphi, -1)}{\sqrt{1+|\nabla\varphi(y)|^2}}$$

$$= n_x.$$

Es decir, $\frac{\partial u}{\partial t}(y,0) = \nabla u(x)\cdot n_x = \frac{\partial u}{\partial n}(x)$.

También,

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\,\frac{\partial^2 x_i}{\partial t^2}$$

$$= D^2 u(x)\, n\cdot n + 0 = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\, n_i n_j,$$

puesto que $\frac{\partial^2 x_i}{\partial t^2} = 0$, si $i = 1, 2, \ldots, n$.

En general,

$$\frac{\partial^\ell u}{\partial n^\ell} = \sum_{i_1, i_2, \ldots, i_\ell = 1}^{n} \frac{\partial^\ell u}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_\ell}}\, n_{i_1} n_{i_2} \cdots n_{i_\ell} = \frac{\partial^\ell u}{\partial t^\ell},$$

si $\ell = 0, 1, 2, \ldots, k-1$ y $k \geq 2$.

Prop: Si $u \in C^{k-1}$ en un entorno de P, conocer

$$u,\ \frac{\partial u}{\partial n},\ \frac{\partial^2 u}{\partial n^2}, \ldots, \text{ y } \frac{\partial^{k-1} u}{\partial n^{k-1}} \text{ en } S,$$

equivale a conocer todas las derivadas

$$\partial_x^\alpha u\big|_S, \quad \text{para } |\alpha| \leq k-1.$$

Dem: En coordenadas (y,t), conocemos

$$u(y,0),\ \partial_t u(y,0), \ldots, \partial_t^{k-1} u(y,0),\ |y-P| \text{ pequeño.}$$

Esto determina $\partial_y^\alpha \partial_t^j u(y,0)$, si $|\alpha| + j \leq k-1$. Deshaciendo el cambio, tenemos $\partial_x^\alpha u|_S$, si $|\alpha| \leq k-1$
