$\Omega \subset \mathbb{R}^n$ abierto.

$$C^\infty(\Omega) = \{\varphi : \Omega \subset \mathbb{R}^n \to \mathbb{R} \,/\, \varphi \text{ es } C^\infty \text{ en } \Omega\}.$$

$$\mathcal{D}(\Omega) = \{\varphi \in C^\infty(\Omega) : \operatorname{supp}(\varphi) \subset \Omega \text{ es compacto}\}.$$

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \sup_{\substack{|\alpha| \le N \\ x \in \mathbb{R}^n}} (1+|x|^2)^N |\partial^\alpha \varphi(x)| < +\infty, \ \forall N \geq 1\}.$$

- $\mathcal{D}(\Omega) \subset C^\infty(\Omega)$

i) $\{\varphi_k\} \to \varphi$ en $\mathcal{D}(\Omega)$ si $\exists\, k \subset \Omega$ compacto tal que $\operatorname{supp}(\varphi_k) \subset k$, $\forall k \geq 1$ y $\partial^\alpha \varphi_k \to \partial^\alpha \varphi$ uniformemente en k, $\forall \alpha \in \mathbb{N}^n$.

ii) $\{\varphi_k\} \to \varphi$ en $C^\infty(\Omega)$ si $\partial^\alpha \varphi_k \to \partial^\alpha \varphi$ uniformemente sobre compactos de Ω, $\forall \alpha \in \mathbb{N}^n$.

iii) $\{\varphi_k\} \to \varphi$ en $\mathcal{S}(\mathbb{R}^n)$, si $\sup_{\substack{|\alpha| \le N \\ x \in \mathbb{R}^n}} (1+|x|)^N |\partial^\alpha \varphi_k(x) - \partial^\alpha \varphi(x)| \to 0$, $\forall N \geq 1$.

- $T \in \mathcal{D}'(\Omega)$ si $T : \mathcal{D}(\Omega) \to \mathbb{R}$ es lineal y $\forall k \subset \Omega$ existen N y $C \geq 1$ tales que

$$|T(\varphi)| = |\langle T, \varphi\rangle| \leq C \sup_{\substack{x \in k \\ |\alpha| \le N}} |\partial^\alpha \varphi(x)|, \quad \forall \varphi \in \mathcal{D}(\Omega), \ \operatorname{supp}(\varphi) \subset k$$

- $T \in C^\infty(\Omega)'$ si $T : C^\infty(\Omega) \to \mathbb{R}$ es lineal y $\exists\, k \subset \Omega$, C y $N \geq 1$ tales que

$$|T(\varphi)| \leq C \sup_{\substack{x \in k \\ |\alpha| \le N}} |\partial^\alpha \varphi(x)|, \quad \forall \varphi \in C^\infty(\Omega).$$

- $T \in \mathcal{S}'(\mathbb{R}^n)$ si existen N y $C \geq 1$ tal que

$$|T(\varphi)| \leq C \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \le N}} (1+|x|)^N |\partial^\alpha \varphi(x)|, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

- Si T es distribución y $\alpha \in \mathbb{N}^n$, $\partial^\alpha T$ es una nueva distribución de la misma clase tal que
$$(\partial^\alpha T, \varphi) = (-1)^{|\alpha|}(T, \partial^\alpha \varphi), \quad \forall \varphi \ldots$$

- $$\mathcal{D}(\Omega) \subset C^\infty(\Omega), \qquad \mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n).$$
$$C^\infty(\Omega)' \subset \mathcal{D}'(\Omega), \qquad C^\infty(\mathbb{R}^n)' \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n).$$

- Si $T \in \mathcal{D}'(\Omega)\ (\mathcal{S}'(\mathbb{R}^n))$, $f \in C^\infty(\Omega)\ (\mathcal{S}(\mathbb{R}^n))$, entonces $fT \in \mathcal{D}'(\Omega)\ (\mathcal{S}'(\mathbb{R}^n))$, siendo
$$(fT, \varphi) = (T, f\varphi), \quad \forall \varphi \in \mathcal{D}(\Omega)\ (\mathcal{S}(\mathbb{R}^n)).$$

- Si $E \in \mathcal{D}'(\mathbb{R}^n)$, $f \in \mathcal{D}(\mathbb{R}^n)$, $E * f \in C^\infty(\mathbb{R}^n)$, donde
$$E * f(x) = (E, f(x - \cdot)), \quad x \in \mathbb{R}^n.$$

- $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x)\,dx$, $\check{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} f(x)\,dx$. Entonces:
  - i) $\wedge : L^1(\mathbb{R}^n) \longrightarrow C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
  - ii) $\wedge : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$. Plancherel: $\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}$ $\forall f \in L^2(\mathbb{R}^n)$. $\vee$ es la inversa de $\wedge$.
  - iii) $\wedge : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, $\check{\hat{f}}(x) = f(x)$, $\forall x \in \mathbb{R}^n$, $f \in \mathcal{S}(\mathbb{R}^n)$.

- Si $T \in \mathcal{S}'(\mathbb{R}^n)$, $\hat{T} \in \mathcal{S}(\mathbb{R}^n)$, siendo
$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle, \quad \text{si } \varphi \in \mathcal{S}(\mathbb{R}^n).$$
Además, $\wedge : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ y $\check{\hat{T}} = T$, si $T \in \mathcal{S}'(\mathbb{R}^n)$.

- $T \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$. $T * \varphi(x) = (T, \varphi(x - \cdot))$, si $x \in \mathbb{R}^n$, $T * \varphi \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ y $\widehat{T * \varphi} = \hat{\varphi}\hat{T}$.

* Si es posible construir $T * \varphi$, $\partial^\alpha(T * \varphi) = (\partial^\alpha T) * \varphi = T * (\partial^\alpha \varphi)$, si $\alpha \in \mathbb{N}^n$.

$$Lu = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha u, \quad a_\alpha \in C^\infty(\Omega)$$

- Si $T, S \in \mathcal{D}'(\Omega)$, $LT = S$ en el sentido de las distribuciones en Ω (solución débil de $LT = S$), si

$$(LT,\varphi) = (T, L^*\varphi) = (S,\varphi), \quad \forall \varphi \in \mathcal{D}(\Omega),$$

Ej

- $f \in C(\mathbb{R})$, $u(x,t) = f(x-ct)$, u verifica $u_t + cu_x = 0$ en $\mathbb{R}^2$ como distribución.
- $T = \chi_{[0,+\infty)}$ en $\mathbb{R}$, $T' = \delta_0$, $\delta_0(\varphi) = \varphi(0)$, $\delta_0 \in C^\infty(\mathbb{R}^n)'$

* Sea $T(x_1,x_2) = \chi_{[0,+\infty)}(x_1)\,\chi_{[0,+\infty)}(x_2)$. Mostrar que

$$\partial^2_{x_1 x_2} T = \delta_0$$

* $u(x) = \frac{1}{2}|x-\xi|$, verifica $u'' = \delta_\xi$, $\delta_\xi(\varphi) = \varphi(\xi)$.

* Mostrar que

$$E(x_1,x_2) = \begin{cases} \frac{1}{2}, & \text{si } |x_1-\xi_1| < \xi_2 - x_2, \\ 0, & \text{en otro caso} \end{cases}$$

es una solución fundamental con polo en (ξ_1,ξ_2) del operador de ondas

$$\square = \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}$$

Def: $L = \sum_{|\alpha|\le k} a_\alpha \partial^\alpha$, $a_\alpha \in \mathbb{C}$. $E \in \mathcal{D}'(\mathbb{R}^n)$ es una solución fundamental para L en $\mathbb{R}^n$, si $LE = \delta_0$. Si L tiene una solución fundamental y $f \in C_0^\infty(\mathbb{R}^n)$, la función,

$$\theta = E * f,$$

está en $C^\infty(\mathbb{R}^n)$ y verifica $L\theta = f$ en $\mathbb{R}^n$.

La solución fundamental no es única,

$\chi_{[0,+\infty)}(x)+\alpha$, $\alpha\in\mathbb{R}$, es también sol. fund. de $\frac{d}{dx}$ en $\mathbb{R}$,

pero lo deseable es encontrar aquella tal que el operador lineal

$$\mathcal{D}(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$$

$$f \longmapsto E * f$$

tenga mejores propiedades.

Tma de Malgrange - Ehrenpreis: $L=\sum_{|\alpha|\le k} a_\alpha \partial^\alpha$, $a_\alpha\in\mathbb{C}$, un operador lineal de coeficientes constantes. Entonces, existe $E\in\mathcal{D}'(\mathbb{R}^n)$ tal que $LE=\delta_0$. Además, si $f\in C_0^\infty(\mathbb{R}^n)$, existe $u\in C^\infty(\mathbb{R}^n)$ tal que $Lu=f$, en $\mathbb{R}^n$.

Ej: Retrocedemos a un ejemplo anterior:

- $E=\frac{1}{2}\chi_\Omega$, $\Omega=\{(x_1,x_2): -x_2>|x_1|\}$, es solución fundamental para $\square=\partial^2_{x_2}-\partial^2_{x_1}$.

Dem: $E\in\mathcal{S}'(\mathbb{R}^2)$

$$|\langle E,\varphi\rangle|\le\int_{\mathbb{R}^2}|\varphi(x_1,x_2)|\,dx_1\,dx_2\le$$

$$\le \sup_{x\in\mathbb{R}^2}(1+|x|)^3|\varphi(x)|\int_{\mathbb{R}^2}\frac{dx}{(1+|x|)^3}$$

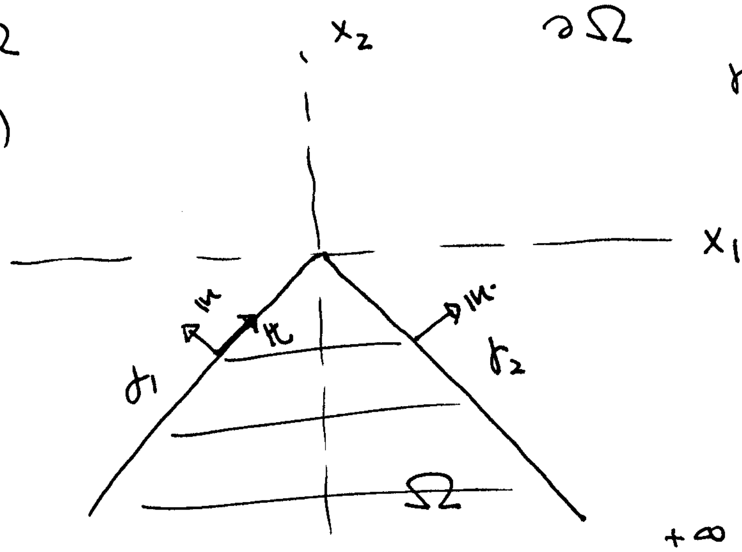
$$\le N\sup_{x\in\mathbb{R}^2}(1+|x|)^3|\varphi(x)|.$$

• Si $\varphi \in \mathcal{S}(\mathbb{R}^4)$,

$$\langle (\partial^2_{x_2}-\partial^2_{x_1}) E, \varphi \rangle = \langle E, (\partial^2_{x_2}-\partial^2_{x_1})\varphi \rangle$$

$$= \frac{1}{2} \int_{\Omega} (\partial^2_{x_2}-\partial^2_{x_1})\varphi \, dx_1 dx_2 = \frac{1}{2}\int_{\partial\Omega} \partial_{x_2}\varphi \, n_2 - \partial_{x_1}\varphi \, n_1 \, d\sigma$$

$n = (n_1, n_2)$



$\gamma_1 \sim y = x, \; -\infty < x < 0$

$n = \frac{(-1,1)}{\sqrt{2}}$

$d\sigma = \sqrt{2}dx$

$\gamma_2 \sim y = -x, \; 0 < x < +\infty$

$n = \frac{(1,1)}{\sqrt{2}}$

$d\sigma = \sqrt{2}dx.$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} \partial_{x_2}\varphi(t,t) + \partial_{x_1}\varphi(t,t)\,dt - \int_0^{+\infty} \partial_{x_1}\varphi(t,-t) - \partial_{x_2}\varphi(t,-t)\,dt \right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} \frac{d}{dt}\varphi(t,t)\,dt - \int_0^{+\infty} \frac{d}{dt}\varphi(t,-t)\,dt \right] = \varphi(0,0)$$

• Si $f \in \mathcal{S}(\mathbb{R}^2)$ y $u(x_1,x_2) = E * f(x_1,x_2)$,

$u \in C^\infty(\mathbb{R}^2) \cap \mathcal{S}'(\mathbb{R}^2)$ y $\square u = f$; pues

$$\square u = E * (\partial^2_{x_2} f - \partial^2_{x_1} f) = [(\partial^2_{x_2}-\partial^2_{x_1})E] * f = \delta_0 * f$$

$$= \langle \delta_0, f(x_1 - \cdot, x_2 - \cdot) \rangle = f(x_1, x_2)$$

$Lu = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha u$, $a_\alpha \in \mathbb{C}$, $f \in C_0^\infty(\mathbb{R}^4)$. Suponemos

$u \in \mathcal{S}'(\mathbb{R}^4) \cap C^\infty(\mathbb{R}^4)$ y que verifica

$$Lu = f, \text{ en } \mathbb{R}^4.$$

Entonces,

$$\widehat{Lu} = \hat{f} = \sum_{|\alpha|\leq k} a_\alpha \widehat{\partial^\alpha u} = \Big[\sum_{|\alpha|\leq k} a_\alpha (2\pi i\xi)^\alpha\Big]\hat{u}$$

$$= P(\xi)\,\hat{u}.$$

Es decir, $$\hat{u} = \frac{1}{P(\xi)}\hat{f}$$

¿ Es $\dfrac{1}{P(\xi)} = \hat{E}$, para alguna $E \in \mathcal{S}'(\mathbb{R}^n)$?

Ej: • $n=1$, $L = \dfrac{\partial^2}{\partial x^2} - 1$, $P(\xi) = -(1+4\pi^2\xi^2)$,

$$-\frac{1}{1+4\pi^2\xi^2} = \hat{E}, \text{ pues } \frac{1}{1+4\pi^2\xi^2} \in L^1(\mathbb{R}). \text{ y}$$

$$E(x) = \int e^{2\pi i\xi x}\frac{1}{1+4\pi^2\xi^2}\,d\xi \quad \text{está en } L^\infty(\mathbb{R}).$$

$$= \int_{\mathbb{R}} \frac{\cos(2\pi\xi x)}{1+4\pi^2\xi^2}\,d\xi$$

• $n=1$, $L = \dfrac{\partial}{\partial x}$, $P(\xi) = 2\pi i\xi$, $\dfrac{1}{2\pi i\xi} \notin \mathcal{S}'(\mathbb{R}^n)$.

¿ Existe $\hat{E}$ distribución en $\mathcal{S}'(\mathbb{R})$ tal que

$$(2\pi i\xi)\hat{E} = 1, \text{ en } \mathcal{S}'(\mathbb{R}) \text{ ?}$$

La respuesta es sí, define para $f \in \mathcal{S}(\mathbb{R})$

$$\langle \hat{E}, f\rangle = \lim_{\varepsilon\to 0} \int_{|\xi|\geq\varepsilon} \frac{f(\xi)}{2\pi i\xi}\,d\xi.$$

Entonces, $\hat{E} \in \mathcal{S}'(\mathbb{R}^n)$ y E estará en $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$

En este caso, es claro que $E = \chi_{[0,+\infty)} + \alpha$, para algún $\alpha \in \mathbb{C}$. Además, por definición

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$$\langle E, f \rangle = \langle \hat{E}, \check{f} \rangle = \lim_{\substack{\varepsilon \to 0^+ \\ R \to +\infty}} \int_{\varepsilon \leq |\xi| \leq R} \frac{\check{f}(\xi)}{2\pi i \xi} \, d\xi$$

$$= \lim_{\substack{\varepsilon \to 0^+ \\ R \to +\infty}} \int_{\varepsilon \leq |\xi| \leq R} \frac{1}{2\pi i \xi} \int_{\mathbb{R}} e^{+2\pi i \xi x} f(x) dx \, d\xi$$

$$= \lim_{\substack{\varepsilon \to 0^+ \\ R \to +\infty}} \int_{\mathbb{R}} f(x) \left(\int_{\varepsilon \leq |\xi| \leq R} \frac{e^{+2\pi i \xi x}}{2\pi i \xi} d\xi \right) dx.$$

$$\int_{\varepsilon \leq |\xi| \leq R} \frac{e^{+2\pi i \xi x}}{2\pi i \xi} d\xi = \int_{\varepsilon \leq |\xi| \leq R} \frac{\sin(2\pi \xi x)}{2\pi \xi} d\xi = \left\{ \begin{array}{l} 2\pi \xi x = t \\ d\xi = \frac{1}{2\pi |x|} dt \end{array} \right\}.$$

$$= \int_{2\pi |x| \varepsilon \leq |t| \leq 2\pi R |x|} \frac{\sin(t)}{(t/x)} \frac{dt}{2\pi |x|} = \frac{\operatorname{sgn}(x)}{2\pi} \int_{2\pi |x| \varepsilon \leq |t| \leq 2\pi R |x|} \frac{\sin(t)}{t} dt.$$

$$\longrightarrow \frac{\operatorname{sgn}(x)}{2\pi} \int_{-\infty}^{+\infty} \frac{\sin t}{t} dt = \frac{1}{2} \operatorname{sgn}(x) = \chi_{[0,+\infty)} - 1/2.$$

Además
$$\left| \int_a^b \frac{\sin(t)}{t} dt \right| \leq N, \qquad \forall (a,b) \subset \mathbb{R}.$$

y por TCD,

$$\langle E, f \rangle = \int_{\mathbb{R}} \frac{1}{2} \operatorname{sgn}(x) f(x) dx = \langle \tfrac{1}{2} \operatorname{sgn}(x), f \rangle.$$

Como $E = \frac{1}{2}\,\mathrm{sgn}(x)$, verifica $E' = \delta_0$. Dada $f \in \mathcal{S}(\mathbb{R})$,

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$$u(x) = E * f(x)$$

$$= \frac{1}{2}\int \mathrm{sgn}(y)\, f(x-y)\,dy = \frac{1}{2}\int \mathrm{sgn}(x-y)\, f(y)\,dy$$

$$= \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right) f(y)\,dy$$

es solución $u \in C^\infty(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$ de $u' = f$.

- Otra forma de encontrar u más general.

Si $f \in \mathcal{D}(\mathbb{R})$, $\hat{f}(\xi + i\eta) = \int e^{-2\pi i x(\xi+i\eta)} f(x)\,dx$,

es holomorfa para $z = \xi + i\eta$ en $\mathbb{C}$. Además,

$$\partial_\xi^\alpha \hat{f}(\xi+i\eta) = \int e^{-2\pi i x(\xi+i\eta)} (-2\pi i x)^\alpha f(x)\,dx,$$

$$(-2\pi i(\xi+i\eta))^\beta\, \partial_\xi^\alpha \hat{f}(\xi+i\eta)$$

$$= (-1)^\beta \int e^{-2\pi i x(\xi+i\eta)}\, \partial_x^\beta\left((-2\pi i x)^\alpha f(x)\right) dx,$$

si $\alpha, \beta \geq 0$. Es decir,

$$(1+|\xi+i\eta|)^N\, |\partial_\xi^\alpha \hat{f}(\xi+i\eta)| \tag{1}$$

$$\leq C_N\, e^{2\pi R|\eta|}\, |B_R| \sup_{\substack{x\in\mathbb{R}\\ |\gamma|\leq N}} (1+|x|)^N\, |\partial^\gamma f(x)|,$$

si $|\alpha| \leq N$.

Si suponemos que $u \in \mathcal{S}'(\mathbb{R})$ y $u' = f$, formalmente

$$\hat{u}(\xi) = \frac{1}{2\pi i \xi} \hat{f}(\xi).$$

por lo que

$$u(x) = \int e^{2\pi i x \xi} \frac{\hat{f}(\xi)}{2\pi i \xi} \, d\xi.$$

pero la integral no tiene sentido. Por ello y teniendo en mente el tma. de Cauchy para funciones holomorfas podemos intentar con

$$u(x) = -\int e^{2\pi i x(\xi + i\eta)} \frac{\hat{f}(\xi + i\eta)}{2\pi i(\xi + i\eta)} \, d\xi \quad , \quad \eta \neq 0.$$

Por el Tma. de Cauchy y (1), $u = u(x)$, no depende de $\eta \neq 0$, $u \in C^\infty(\mathbb{R})$,

$$u'(x) = -\int_{\mathbb{R}} e^{2\pi i x(\xi + i\eta)} \hat{f}(\xi + i\eta) \, d\xi \qquad (2)$$

$$= \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi) \, d\xi = f(x), \text{ en } \mathbb{R}.$$

y (2) es consecuencia del Tma. de Cauchy

Para encontrar $E \in \mathcal{D}'(\mathbb{R})$ tal que $E' = \delta_0$, observamos que para $\varphi \in \mathcal{D}(\mathbb{R})$, se verifica que

$$\widehat{\varphi'} = (2\pi i \xi)\, \hat{\varphi} = \hat{f}, \quad \hat{\varphi} = \frac{1}{2\pi i \xi}\hat{f}$$ (10)

$$\varphi(x) = \int e^{2\pi i x \xi}\, \hat{\varphi}(\xi)\, d\xi = -\int e^{2\pi i x(\xi + i\eta)}\, \hat{\varphi}(\xi + i\eta)$$

$$= -\int e^{2\pi i x(\xi+i\eta)} \frac{\widehat{\varphi'}(\xi+i\eta)}{2\pi i(\xi+i\eta)}\, d\xi, \quad \forall \eta \neq 0$$

$$\therefore \varphi(0) = -\int \frac{\widehat{\varphi'}(\xi+i)}{2\pi i(\xi+i)}\, d\xi \qquad (3)$$

Define:

$$\langle E, f\rangle = \int \frac{\hat{f}(\xi+i)}{2\pi i(\xi+i)}\, d\xi,$$

para $f \in \mathcal{D}(\mathbb{R})$. Entonces, $E \in \mathcal{D}'(\mathbb{R})$ y

$\langle E', \varphi\rangle = -\langle E, \varphi'\rangle = \varphi(0)$, por (3), si $\varphi \in \mathcal{D}(\mathbb{R})$. $\therefore$ $E' = \delta_0$, en sentido distribucional.

Para comprobar que $E \in \mathcal{D}'(\mathbb{R})$,

$$|\langle E, f\rangle| \leq \frac{1}{2\pi}\int \frac{|\hat{f}(\xi+i)|}{\sqrt{1+\xi^2}}\, d\xi,$$

pero si $\operatorname{supp}(f) \subset B_R$, $\displaystyle |\hat{f}(\xi+i)| \leq \frac{N}{|\xi+i|}\int_{\mathbb{R}} e^{2\pi|x|}\, |f'(x)|\, dx$

$$\leq \frac{N}{|\xi+i|}\, e^{2\pi R}\, R \sup_{B_R} |f'|; \quad \text{es decir,}$$

$$\int_{\mathbb{R}} \frac{|\hat{f}(q+i)|}{\sqrt{1+q^2}}\, dq \leq \left(\int_{\mathbb{R}} \frac{dq}{1+q^2}\right) e^{4\pi R} \sup_{B_R} |f'|;$$

que implica, $E \in \mathcal{D}'(\mathbb{R})$.

Antes he afirmado que existe $N>0$ tal que

$$\left|\int_a^b \frac{\sin t}{t}\, dt\right| \leq N, \quad \forall (a,b) \subset \mathbb{R}.$$

Dem: Como $\frac{\sin t}{t}$ es par, podemos suponer $0 \leq a < b$.

Entonces, si $(a,b) \subset (0,1)$,

$$\left|\int_a^b \frac{\sin t}{t}\, dt\right| \leq \int_0^1 \left|\frac{\sin t}{t}\right| dt.$$

Si $0 < a \leq 1$, $b \geq 1$,

$$\left|\int_a^b \frac{\sin t}{t}\, dt\right| \leq \int_0^1 \left|\frac{\sin t}{t}\right| dt + \left|\int_1^b \frac{\sin t}{t}\, dt\right|. \qquad (4)$$

Para $1 \leq a < b$,

$$\int_a^b \frac{\sin t}{t}\, dt = \int_a^b \frac{(-\cos t)'}{t}\, dt = -\frac{\cos t}{t}\Big]_a^b - \int_a^b \frac{\cos t}{t^2}\, dt$$

$$= -\frac{\cos b}{b} + \frac{\cos a}{a} - \int_a^b \frac{\cos t}{t^2}\, dt. \text{ Es decir,}$$

$$\left|\int_a^b \frac{\sin t}{t}\, dt\right| \leq 2 + \int_1^{+\infty} \frac{dt}{t^2} = 3. \qquad (5)$$

(4) y (6) implican lo afirmado.

La solución fundamental para Δ

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \quad \text{Si} \quad u \in \mathcal{S}(\mathbb{R}^n) \text{ y } \Delta u = f,$$

$$-4\pi^2|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi), \ \forall \xi \in \mathbb{R}^n \text{ y}$$

$$\hat{u}(\xi) = -\frac{1}{4\pi^2|\xi|^2} \hat{f}(\xi) \quad \text{si} \quad \xi \neq 0.$$

Si $n \geq 3$, $\frac{-1}{4\pi^2|\xi|^2} \in \mathcal{S}'(\mathbb{R}^n)$, pues

$$\left| \int_{\mathbb{R}^n} \frac{1}{|\xi|^2} \varphi(\xi)\, d\xi \right| \leq \int \frac{1}{|\xi|^2} |\varphi(\xi)|\, d\xi \leq \left(\int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^2} \right) \sup_{\mathbb{R}^n} |\varphi|$$

$$+ \left(\int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{n+1}} \right) \sup_{\mathbb{R}^n} |\xi|^{n-1} |\varphi(\xi)|$$

$$\leq \left(\int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^2} + \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{n+1}} \right) \sup_{\mathbb{R}^n} (1+|\xi|)^{n-1} |\varphi(\xi)|,$$

y existe $E \in \mathcal{S}'(\mathbb{R}^n)$ tal que $\hat{E} = \frac{-1}{4\pi^2|\xi|^2}$.

¿Quién es E? Si $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle E, \varphi \rangle = \langle \hat{E}, \check{\varphi} \rangle = -\frac{1}{4\pi^2} \int_{\mathbb{R}^n} |\xi|^{-2} \check{\varphi}(\xi)\, d\xi.$$

$$= \lim_{R \to +\infty} \int_{\mathbb{R}^n} \varphi_R(\xi) |\xi|^{-2} \check{\varphi}(\xi)\, d\xi, \quad \varphi_R(\xi) = \varphi\left(\frac{\xi}{R}\right), \quad (6)$$

$\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$, $|\xi| \leq 1$, $\varphi = 0$, si $|\xi| \geq 2$.

$$= \lim_{R \to +\infty} \int_{\mathbb{R}^n} \varphi(x) \left(\int_{\mathbb{R}^n} e^{2\pi i x \xi} \varphi_R(\xi) |\xi|^{-2} d\xi \right) dx$$

$$= \lim_{R \to +\infty} \int_{\mathbb{R}^n} \varphi(x) T_R(x)\, dx, \qquad T_R(x) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} \varphi_R(\xi) |\xi|^{-2} d\xi.$$

Veremos que

- $$\lim_{R \to +\infty} T_R(x) = \frac{D_n}{|x|^{n-2}}, \quad \text{si } x \neq 0 \qquad (7)$$
- $$|T_R(x)| \leq \frac{N_n}{|x|^{n-2}}, \quad \text{si } x \neq 0, \ R > 0.$$

y de (6), (7) y el TCD se sigue que

$$\langle E, \varphi \rangle = \int_{\mathbb{R}^n} \frac{D_n}{|x|^{n-2}} \varphi(x)\, dx, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Para comprobar (7),

$$T_R(x) = \left\{ \begin{array}{l} \xi = \dfrac{\eta}{|x|} \\ d\xi = |x|^{-n} d\eta \end{array} \right\} = |x|^{2-n} \int_{\mathbb{R}^n} e^{2\pi i \frac{x}{|x|} \cdot \xi} \varphi_{R|x|}(\xi) |\xi|^{-2} d\xi$$

$$= \left\{ \begin{array}{ll} Q \text{ rotación de } \mathbb{R}^n, & \xi = Q\eta \\ Q^t \dfrac{x}{|x|} = e_1, \ Q^t = Q^{-1}, & d\xi = d\eta \\ |\det Q| = 1, \ |Q\xi| = |\xi|, & \end{array} \right\}$$

$$= |x|^{2-n} \int_{\mathbb{R}^n} e^{2\pi i \xi_1} \varphi_{R|x|}(\xi) |\xi|^{-2} d\xi,$$

con $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \equiv 1$ en B_1, $\varphi \equiv 0$ fuera de B_2. y (7) es consecuencia de los siguientes hechos:

• Existe N_n tal que $\left| \int_{\mathbb{R}^n} e^{i\xi_1} \varphi_R(\xi) |\xi|^{-2} d\xi \right| \leq N_n$ (8)

para todo $R \geq 1$.

* Existe $D_n = \lim_{R \to +\infty} \int e^{i\xi_1} \varphi_R(\xi) |\xi|^{-2} d\xi$.

Para comprobar (8), si $R \leq 1$

$$\left| \int e^{i\xi_1} \varphi\left(\frac{\xi}{R}\right) |\xi|^{-2} d\xi \right| \leq \int_{B_1} |\xi|^{-2} d\xi \leq N_n.$$

Si $R \geq 1$,

$$\int e^{i\xi_1} \varphi_R(\xi) |\xi|^{-2} d\xi = \int_{|\xi| \leq 1} e^{i\xi_1} |\xi|^{-2} d\xi + \int_{|\xi| \geq 1} e^{i\xi_1} |\xi|^{-2} \varphi_R(\xi) d\xi$$

y si $n=3$,

$$\int_{|\xi| \geq 1} e^{i\xi_1} |\xi|^{-2} \varphi_R(\xi) d\xi = - \int_{|\xi| \geq 1} \partial^2_{\xi_1} e^{i\xi_1} |\xi|^{-2} \varphi_R(\xi) d\xi.$$

$$= - \int_{|\xi| = 1} \left[(\partial_{\xi_1} e^{i\xi_1}) |\xi|^{-2} - e^{i\xi_1} \partial_{\xi_1} (|\xi|^{-2}) d\xi \right] d\sigma$$

$$- \int_{|\xi|\geq 1} e^{i\xi_1} \partial^2_{\xi_1} \left(|\xi|^{-2} \varphi_R(\xi)\right) d\xi.$$

La última integral es igual a

$$\int_{|\xi|\geq 1} e^{i\xi_1} \left(\partial^2_{\xi_1} |\xi|^{-2}\right) \varphi_R(\xi)\, d\xi + 2 \int_{|\xi|\geq 1} e^{i\xi_1} \partial_{\xi_1} |\xi|^{-2}\, \partial_{\xi_1} \varphi_R(\xi)\, d\xi$$

$$+ \int_{|\xi|\geq 1} e^{i\xi_1} |\xi|^{-2} \partial^2_{\xi_1} \varphi_R(\xi)\, d\xi.$$

Por TCD, $\displaystyle\int_{|\xi|\geq 1} e^{i\xi_1} \left(\partial^2_{\xi_1} |\xi|^{-2}\right) \varphi_R(\xi)\, d\xi \to \int_{|\xi|\geq 1} e^{i\xi_1} \partial^2_{\xi_1} \left(|\xi|^{-2}\right) d$

una integral absolutamente convergente y las dos últimas integrales verifican:

$$\left| \int_{|\xi|\geq 1} e^{i\xi_1} \left[2\partial_1 |\xi|^{-2} \partial_{\xi_1} \varphi_R(\xi) + |\xi|^{-2} \partial^2_{\xi_1} \varphi_R(\xi) \right] d\xi \right|$$

$$\lesssim \int_{R \leq |\xi| \leq 2R} R^{-4}\, d\xi \lesssim \frac{1}{R} \to 0, \text{ si } R \to +\infty.$$