Books, problem sets and files sent by Email

W. Rudin. *Real and Complex Analysis*.pdf

W. Rudin. *Functional Analysis*.pdf

H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.pdf

K. Saxe. *Beginning Functional Analysis, Beginning Functional Analysis*.pdf

S. Semmes. *An introduction to some aspects of functional analysis*.pdf

D. Gilbarg, N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*.pdf

Lecture Notes Functional Analysis.pdf

General guidelines.pdf

To move faster over the book files, use the following:

Add up 13 in W. Rudin. *Real and Complex Analysis*.pdf. The index is in p. 6.

Add up 15 in W. Rudin. *Functional Analysis*.pdf. The index is in p. 6

Add up 9 in H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.pdf. The index is in p. 8.

Add up 11 in K. Saxe. *Beginning Functional Analysis*.pdf. The index is in p. 9 and the bibliography in p. 198

Problem sets

Choose 4 problems from each problem set below, solve and turn them in on the final exam day. Please, write and staple together the 4 problems of each problem set one after the other by problem sets, with your name, surnames and problem set number in the first page of each group.

Everybody must turn in its hand or Latex written solutions to the later problems on the final exam day. Finally, it is mandatory for everybody to turn in the solution to problem 6.3.7 within your choices of problems from the 6th problem set.

Chapter 1: 1.1.8, 1.1.9, 1.3.2, 1.3.3

Chapter 2: 2.1.9, 2.1.12, 2.1.13, 2.2.1, 2.2.2, 2.2.3, 2.3.5, 2.3.6, 2.3.7, 2.3.8

Chapter 3: 3.6.9

Chapter 4: 4.1.1, 4.1.2, 4.2.1, 4.1.4, 4.1.7, 4.2.3, 4.2.4, 4.2.6, 4.2.7

Chapter 5: 5.1.3, 5.2.1, 5.2.2, 5.2.3, 5.2.4, 5.2.5, 5.2.6, 5.2.7, 5.3.1, 5.3.3, 5.3.11, 5.3.12, 5.3.13, 5.3.14, 5.4.1, 5.4.4, 5.4.6, 5.4.8, 5.4.9, 5.4.10, 5.5.2

Chapter 6: 6.3.1, 6.3.2, 6.3.3, 6.3.4, 6.3.5, 6.3.6, 6.3.7, 6.3.8

Problems added to the problem sets you will not find in the book

4.2.6 Let H be a Hilbert space and $\mathcal{F}$ be the set of all the orthonormal families in H. Show that the inclusion is an order relation in $\mathcal{F}$. From Zorn's Lemma show that H has a complete orthonormal basis $\{e_i\}_{i\in I}$ for some set of indices I. Then, derive that H is linearly isometric to $l^2(I) = L^2(I, \mathcal{P}(I), dc)$.

4.2.7 Let H be a Hilbert space. Prove that H is separable if and only if H has a countable orthonormal basis.

5.4.8 Prove the properties stated about the adjoint operator T^* of an operator T in $\mathcal{B}(H)$ in p. 22 and in Theorem 5.17, p. 23 of *Lecture Notes*.pdf. This version of Theorem 5.17 replaces the corresponding one in the book.

5.4.9 Prove the properties stated in the two *Examples* in p. 23 of *Lecture Notes*.pdf. When are those operators self-adjoint? Are they compact operators?

5.4.10 Prove that in a non separable Hilbert space all T in $\mathcal{B}(H)$ has a nontrivial invariant subspace.

6.3.3 Let X be a normed space and $x \in X$. Show that

$$\|x\| = \sup\{|\Lambda(x)| : \Lambda \in X^*, \|\Lambda\| \le 1\} = \max\{|\Lambda(x)| : \Lambda \in X^*, \|\Lambda\| \le 1\}.$$

6.3.4 Prove that the dual of l^p is $l^{p'}$, when $1 \le p < \infty$.

6.3.5 Show that the spaces

$$c_0 = \{x \in l^\infty : \lim_{n\to\infty} x_n = 0\} \quad \text{and} \quad c = \{x \in l^\infty : \exists \lim_{n\to\infty} x_n \in \mathbb{C}\}$$

are closed in l^∞. Give examples of Schauder basis for the spaces l^p, $1 \le p < \infty$, c_0 and c.

6.3.6 Prove that every normed space with a Schauder basis is separable. Can l^∞ have a Schauder basis?

6.3.7 Prove using diagonalization that any separable Hilbert space is weakly compact; i.e. any bounded sequence $\{x_n\}$ in H has a subsequence $\{x_{n_k}\}$, which converges weakly to some x in H. Finally, show that the same result holds for any Hilbert space (See theorem 5.12 in D. Gilbarg, N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order* but be careful because the proof there is not completely correct)

6.3.8 Let X be a complex Banach space and F a closed subspace of X. Prove that the quotient space

$$X/F = \{x + F : x \in X\}$$

is a complex vector space. Define

$$\|x + F\| = d(x, F) = \inf_{y\in F} \|x - y\|$$

and show that it defines a norm in X/F. Show that the mapping $\pi : X \longrightarrow X/F$, with $\pi(x) = x + F$, when $x \in X$, is a linear bounded mapping from X onto X/F. Finally, show that X/F is a Banach space (Hint: use the characterization of complete normed spaces in terms of absolutely convergent series)

Duals of L^p spaces for $1 \leq p < \infty$

1. Read the definition of $(X, \mathcal{M}, \mu)$ in p. 97 of H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.pdf.

2. Read theorem 4.11 in p. 105 of H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.pdf; the Riesz representation theorem for the dual of $L^p(X, \mathcal{M}, \mu)$: when $1 < p < \infty$ and μ is a σ-finite measure

$$L^p(X, \mathcal{M}, \mu)^* \cong L^{p'}(X, \mathcal{M}, \mu),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

3. Read theorem 4.14 in p. 107 of H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.pdf: the Riesz representation theorem for the dual of $L^1(X, \mathcal{M}, \mu)$, with μ a σ-finite measure.

4. Read also theorem 6.16 in p. 140 of W. Rudin. *Real and Complex Analysis.* There, X is a measure space $(X, \mathcal{M}, \mu)$ with a σ-finite measure μ over the σ-algebra $\mathcal{M}$.

Duals of l^∞ and L^∞

1. We have shown that $\left(l^1\right)^* = l^\infty$. To verify that $l^1 \subsetneq (l^\infty)^*$ extend by Hahn-Banach to l^∞ the linear functional $f : c \longrightarrow \mathbb{C}$, with $f(x) = \lim_{n\to\infty} x_n$, when $x \in c$. Then, show that $f \neq f_y$, for all $y \in l^1$, where $f_y(x) = \sum_{n=1}^{+\infty} x_n \overline{y_n}$.

2. Read Remark 7 in p. 110 of H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.pdf. It should convince you that one cannot understand a first description of the dual of $L^\infty(X, \mathcal{M}, \mu)$, without knowing more Functional Analysis and Abstract Topology.

2. We have seen that $l_1 \subsetneq l_\infty^*$. Generally, $L^1(X, \mathcal{M}, \mu) \subsetneq L^\infty(X, \mathcal{M}, \mu)^*$. When $X = K$ in $\mathbb{R}^n$ and $0 \in K$, extend by Hahn-Banack to $L^\infty(K)$ the bounded linear functional $\Lambda : C(K) \longrightarrow \mathbb{C}$, with $\Lambda(\varphi) = \varphi(0)$, when $\varphi \in C(K)$. One can verify that $\Lambda \neq \Lambda_g$, for all $g \in L^1(K)$, where $\Lambda_g(f) = \int_K f(x)\overline{g}(x)\, dx$.

3. In pp. 21-23 of S. Semmes. *An introduction to some aspects of functional analysis*.pdf there is a reasonable description of the dual of $L^\infty(X, \mathcal{M}, \mu)$.

Duals of $C_0(X)$ and $C(X)$, when $X = \Omega$ o K

1. We follow the notation in W. Rudin. *Real and Complex Analysis*.pdf, where X stands for a Hausdorff locally compact space; as $X = K$, a compact set in $\mathbb{R}^n$ o $X = \Omega$ an open set in $\mathbb{R}^n$.

$C_b(X)$ denotes the space of complex-valued bounded continuous functions over X with the sup norm; i.e.

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

$C_b(X)$ is a Banach space, as $C(K)$.

2. Read the definition 2.9 for $C_c(X)$ in p. 51 of W. Rudin. *Real and Complex Analysis*.pdf.

3. Read the definition 3.16 of $C_0(X)$ and theorem 3.17 in p. 83 of W. Rudin. *Real and Complex Analysis*.pdf. Observe that for $X = K$, $C_c(K) = C_0(K) = C(K) = C_b(K)$.

4. Read the definitions in pp. 129 and 130 of W. Rudin. *Real and Complex Analysis*.pdf (the introduction of chapter 6), theorem 6.2 in p. 130, theorem 6.4 in p. 131, the definitions 6.5 and 6.6 in p. 132, the theorems 6.12 in p. 137 and 6.13 in p. 138.

5. Finally, read the definitions in 6.18 and theorem 6.19 in pp. 142 and 143.

6. I recommend to read the later definitions and theorems so that you can understand the statement of the Riesz representation theorem below (another version. . .):

Theorem 1 (Riesz representation theorem)**.** *Let $X = \Omega$ or $X = K$ en $\mathbb{R}^n$. Then, the dual of $C_0(X)$ is $\mathcal{M}(X)$, where $\mathcal{M}(X)$ is the space of all complex-valued measures $\mu : \mathcal{B}_X \longrightarrow \mathbb{C}$, with norm $\|\mu\| = |\mu|(X)$, where $|\mu|$ is the total variation of μ and $\mathcal{B}_X$ the σ-algebra of all Borel sets in X. In particular, the mapping*

$$\mathcal{M}(X) \longrightarrow C_0(X)^*, \ \mu \rightsquigarrow \Lambda_\mu,$$

with

$$\Lambda_\mu(f) = \int_X f(x)\, d\mu(x), \quad \textit{when } f \in C_0(X),$$

is a linear one-to-one isometry from $\mathcal{M}(X)$ onto $C_0(X)^*$.

Examples of non separable normed spaces

We have shown that l^∞ and $L^\infty(0,1)$ are non separable. Generally, neither $L^\infty(K)$ or $L^\infty(\Omega)$ are separable. The later can be verified with similar reasonings. On the contrary, l^p, $L^p(0,1)$, $L^p(K)$ and $L^p(\Omega)$, $1 \le p < +\infty$, are all separable.

l^∞ and $L^\infty(0,1)$ are non separable because the families

$$\{x \in l_\infty : x_n = 0 \text{ o } 1, \text{ for all } n \ge 1\} \subset l^\infty \quad \text{and} \quad \{\chi_{(0,a)} : 0 < a < 1\} \subset L^\infty(0,1),$$

have both cardinal c and the distance between two distinct elements within each family in l_∞ and $L^\infty(0,1)$ is always greater or equal than 1.

Other non separable Banach space is $C^\alpha([0,1])$, $0 < \alpha \le 1$, the space of Hölder continuous functions with exponent α in $[0,1]$, with norm

$$\|f\| = \|f\|_{L^\infty([0,1])} + \sup_{t \ne s} \frac{|f(t) - f(s)|}{|t-s|^\alpha}.$$

The family of functions $\{f_a : 0 < a \le 1\} \subset C^\alpha([0,1])$ with

$$f_a(x) = \max\{0, x-a\}^\alpha,$$

has cardinal number c and $\|f_a - f_b\| \ge 1$, when $a \ne b$, because in such case

$$\frac{|(f_a - f_b)(b) - (f_a - f_b)(a)|}{|b-a|^\alpha} = 1,$$

when $a < b$.

Other interesting results on Functional Analysis we sketched out during the lectures:

Reflexive spaces

When $(X, \|\cdot\|)$ is a norm space, the mapping

$$i : X \longrightarrow X^{**}/x \rightsquigarrow i_x, \text{ with } i_x(\Lambda) = \Lambda(x), \text{ for } \Lambda \in X^*, \tag{0.1}$$

is linear and continuos. The Hahn-Banach theorem shows that it is a one-to-one isometry from X onto its image inside X^{**} i.e.

$$\|x\|_X = \|i_x\|_{X^{**}}, \text{ for all } x \in X. \tag{0.2}$$

To verify (0.2), recall that

$$\begin{aligned}\|i_x\|_{X^{**}} &= \sup\{|i_x(\Lambda)| : \Lambda \in X^*, \ \|\Lambda\| \le 1\} \\ &= \sup\{|\Lambda(x)| : \Lambda \in X^*, \ \|\Lambda\| \le 1\}.\end{aligned}$$

Definition 1. A Banach space X is called reflexive when the mapping i in (0.1) is onto; i.e., it is a linear one-to-one and onto isometry between X and X^{**}.

Let H be a Hilbert space. By the Riesz-Fréchet representation theorem, the mapping $\psi : H \longrightarrow H^*$, $y \rightsquigarrow \psi_y$, with $\psi_y(x) = \langle x, y\rangle$, when $x \in H$, is a linear conjugate one-to-one and onto mapping . One can then endow H^* with a Hilbert space structure by defining in $H^* \times H^*$, the dot product

$$\langle \psi_{y_1}, \psi_{y_2}\rangle_{H^*} = \langle y_2, y_1\rangle, \text{ when } \psi_{y_1}, \psi_{y_2} \in H^*. \tag{0.3}$$

Observe that this product yields the original norm in H^*; i.e.

$$\|\psi_y\| = \sqrt{\langle \psi_y, \psi_y\rangle_{H^*}}, \text{ when } \psi_y \in H^* .$$

Analogously, $\varphi : H^* \longrightarrow H^{**}$, with $\Lambda \rightsquigarrow \varphi_\Lambda$, where

$$\varphi_\Lambda(\psi_x) = \langle \psi_x, \Lambda\rangle_{H^*}, \text{ when } \psi_x \in H^*,$$

is a linear conjugate one-to-one and onto mapping. It is easy to verify that i in (0.1) with $X = H$ is the same a $\varphi \circ \psi$. Finally, the composition of two linear conjugate one-to-one and onto isometries is a linear one-to-one and onto isometry. The later shows that all Hilbert spaces are all reflexive.

Other examples of reflexive Banach spaces are $L^p(X, \mathcal{M}, \mu)$, when $1 < p < \infty$ and $(X, \mathcal{M}, \mu)$ is a measure space with a σ-finite measure. This follows from the Riesz representation theorem for the dual of $L^p(X, \mathcal{M}, \mu)$; i.e.

$$L^p(X, \mathcal{M}, \mu)^* \cong L^{p'}(X, \mathcal{M}, \mu),$$

the fact that the mapping

$$\Gamma : L^{p'}(X, \mathcal{M}, \mu) \longrightarrow L^p(X, \mathcal{M}, \mu)^*, \text{ with } g \rightsquigarrow \Gamma_g,$$

where

$$\Gamma_g(f) = \int_X f\overline{g}\, d\mu,$$

is a linear conjugate one-to-one and onto mapping and from similar reasonings to the ones above: use twice the Riesz representation theorem.

The l^p spaces, $1 < p < \infty$ are also reflexive. To check it use the same ideas and the fact that $(l^p)^* \approx l^{p'}$, when $1 \le p < \infty$.

On the contrary, l^1 and $L^1(\Omega)$ are not reflexive because its duals are l_∞ and $L^\infty(\Omega)$ respectively, while the duals of the later are larger than l^1 and $L^1(\Omega)$; i.e., the imbeddings i in (0.1) are not onto, when X is either l^1 o $L^1(\Omega)$.

Another example of a non-reflexive Banach space is $C^\alpha([0,1])$, when $0 < \alpha \le 1$.

Definition 2. A sequence $\{x_n\}$ in a Hilbert space X converges weakly to $x \in X$, when $\lim_{n\to+\infty} \Lambda(x_n) = \Lambda(x)$, for all Λ in X^*.

Definition 3. A sequence $\{\Lambda_n\}$ in X^* converges weak-$*$ to $\Lambda \in X^*$, when

$$\lim_{n\to+\infty} \Lambda_n(x) = \Lambda(x), \text{ for all } x \in X^*.$$

The later convergences are weaker than the corresponding norm convergences; i.e., if $\{x_n\}$ ($\{\Lambda_n\}$) converges in norm to x (Λ) in X (X^*), then $\{x_n\}$ ($\{\Lambda_n\}$) converges weakly (weak-$*$) to x (Λ) in X (X^*).

The reciprocal is false: by the Riemann-Lebesgue lemma, the sequence $\{e^{inx}\}$ converges weakly to zero in $L^p([-\pi,\pi])$, when $1 \le p < \infty$; it also converges weak-$*$ to zero in $\mathcal{M}([-\pi,\pi]) = C([-\pi,\pi])^*$ and in $L^\infty(-\pi,\pi) = L^1(-\pi,\pi)^*$, but it does not converge to zero in these spaces. Observe that

$$\|e^{inx}\|_{L^p(-\pi,\pi)} = (2\pi)^{\frac{1}{p}}, \text{ when } 1 \le p \le \infty$$

and that the total variation of $e^{inx}\,dx$ in $[-\pi,\pi]$ is Lebesgue measure, whose total variation norm is 2π in $\mathcal{M}([-\pi,\pi])$.

Theorem 2 (Banach-Alaoglu theorem for reflexive spaces)**.** *Let X be a reflexive norm space. Then, every bounded sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$, $n_1 < n_2 < \cdots < n_k < \ldots$, converging weakly to some x in X; i.e.,*

$$\lim_{k\to+\infty} \Lambda(x_{n_k}) = \Lambda(x), \textit{ for all } \Lambda \in X^*.$$

Theorem 3 (Banach-Alaoglu theorem)**.** *Let X be a Banach space. Then, every bounded sequence $\{\Lambda_n\}$ in X^* has a subsequence $\{\Lambda_{n_k}\}$, $n_1 < n_2 < \cdots < n_k < \ldots$, converging weak-$*$ to some Λ in X^*; i.e.,*

$$\lim_{k\to+\infty} \Lambda_{n_k}(x) = \Lambda(x), \textit{ for all } x \in X.$$

Theorem 3 implies theorem 2 because reflexive spaces X are linearly isometric to X^{**}; i.e., essentially equal to X^{**}.

In particular, Hilbert spaces H are reflexive and when $\{x_n\}$ is a bounded sequence in H, there are $n_1 < n_2 < \cdots < n_k \ldots$ and x in H, such that the subsequence $\{x_{n_k}\}$ verifies

$$\lim_{k\to+\infty} \langle x_{n_k}, y\rangle = \langle x, y\rangle, \text{ for all } y \in H.$$

The last two theorems and the last example show that the best posible replacement of the Heine-Borel property for finite dimensional normed spaces within the context of infinite dimensional Banach spaces are theorems 2 and 3; i.e.: when X is infinite dimensional, theorems 2 and 3 provide the best that one can say about the possible convergence of subsequences of a bounded sequences in X^*!

Moreover, most norm spaces are duals or are contained in the dual of some Banach space. For instance, $L^p(\Omega)^* \cong L^{p'}(\Omega)$, when $1 \le p < \infty$ and $L^1(K) \subsetneqq \mathcal{M}(K) = C(K)^*$, by the Riesz representation theorem 1.

You can find information about the Banach-Alaoglu theorem in §3.15, p. 85 of W. Rudin. *Functional Analysis*.pdf.

An important characterization of reflexive spaces is the following:

Theorem 4 (Kakutani theorem)**.** *A Banach space X is reflexive if and only if the closed unit ball B_X is compact in X for the weak topology in X.*

There is a proof of this result in p. 76 of H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.pdf