(2) $F: \mathbb{X} \longrightarrow \mathbb{X}'$

- Si $\mathcal{A}$ es σ-álgebra en $\mathbb{X}$, entonces $\mathcal{A}' = \{ B \subset \mathbb{X}' : F^{-1}(B) \in \mathcal{A} \}$ es σ-álgebra en $\mathbb{X}'$.
- Si $\mathcal{A}'$ es σ-álgebra en $\mathbb{X}'$, entonces $F^{-1}(\mathcal{A}) = \{ F^{-1}(B) : B \in \mathcal{A}' \}$ es σ-álgebra en $\mathbb{X}$.
- Si $\mathcal{E} \subset \mathcal{P}(\mathbb{X}')$, entonces $\sigma(F^{-1}(\mathcal{E})) = F^{-1}(\sigma(\mathcal{E}))$

Proof. $F^{-1}(\sigma(\mathcal{E}))$ es una σ-álgebra en $\mathbb{X}$ que contiene a $F^{-1}(\mathcal{E})$, luego $\sigma(F^{-1}(\mathcal{E})) \subset F^{-1}(\sigma(\mathcal{E}))$. Si $\mathcal{A}$ es una σ-álgebra en $\mathbb{X}$ que contiene a $F^{-1}(\mathcal{E})$, entonces su $\mathcal{A}'$ contiene a $\mathcal{E}$ y por def. $\sigma(\mathcal{E}) \subset \mathcal{A}'$. Entonces, $F^{-1}(\sigma(\mathcal{E})) \subset F^{-1}(\mathcal{A}')$ y $F^{-1}(\mathcal{A}') = \{ F^{-1}(B) : B \in \mathcal{A}' \} \subset \mathcal{A}$. Es decir, $F^{-1}(\sigma(\mathcal{E})) \subset \mathcal{A}$ y $F^{-1}(\sigma(\mathcal{E})) = \sigma(F^{-1}(\mathcal{E}))$.

(1) $\mathbb{X} = \{a, b, c, d\}$, $\mathcal{A} = \{\{a\}, \{b\}\}$. ¿$\sigma(\mathcal{A})$?

$\sigma(\mathcal{A}) = \{ \phi, \mathbb{X}, \{a\}, \{b, c, d\}, \{b\}, \{a, c, d\}, \{a, b\}, \{c, d\}$

(5) (b) $\{n\} \in \mathcal{H}_{A_n}$, $\forall n \geq 1$. Si $\bigcup_{n=1}^{\infty} \mathcal{H}_{A_n}$ fuera σ-álgebra, entonces $\mathbb{N} \in \mathcal{H}_{A_i}$ para algún i y esto es falso.

(6) Podemos suponer que $\phi \in \mathcal{E}$, pues $\mathcal{E}$ y $\mathcal{E} \cup \{\phi\}$ generan la misma álgebra.

- $\phi \in \mathcal{E}_1 \Rightarrow \phi \in \mathcal{E}_3$
  Complementarios de $\mathcal{E}_1$'s están en $\mathcal{E}_1$.
  Complementarios de $\mathcal{E}_2$'s están en $\mathcal{E}_3$.
- Si $A, B \in \mathcal{E}_3$, entonces $A \cup B \in \mathcal{E}_3$
- Si $A, B \in \mathcal{E}_2$, $(A \cup B)^c = A^c \cap B^c = (A_1 \cap A_2)^c \cap (B_1 \cap B_2)^c$
  $= (A_1^c \cup A_2^c) \cap (B_1^c \cup B_2^c) = A_1^c \cap B_1^c \cup A_1^c \cap B_2^c \cup A_2^c \cap B_1^c$
  $\cup A_2^c \cap B_2^c$ y $A_i^c, B_i^c \in \mathcal{E}_1$, las intersecciones a $\mathcal{E}_2$.

En general, si $A_1, \dots, A_m \in \mathcal{C}_2$,

$$A_i = \bigcap_{j=1}^{N} A_{ij}, \quad A_{ij} \in \mathcal{C}_1.$$

$$A_i^c = \bigcup_{j=1}^{N} A_{ij}^c, \quad A_{ij}^c \in \mathcal{C}_1$$

$$(A_1 \cup \dots \cup A_m)^c = \bigcap_{i=1}^{m} A_i^c = \bigcap_{i=1}^{m} \bigcup_{j=1}^{N} A_{ij}^c =$$

$$= \bigcup_{j=1}^{N} \underbrace{\bigcap_{i=1}^{m} A_{ij}^c}_{\in\, \mathcal{C}_2}, \text{ pues } \mathcal{C}_1 \text{ es cerrado bajo complemento}$$

(7b) $\mathcal{C}$ clase de conjuntos finitos. $\sigma(\mathcal{C}) = \mathcal{A} = \{E \subset X : E$ o E^c es numerable, luego numerable$\}$.

$\mathcal{C} \subset \mathcal{A}$; luego $\sigma(\mathcal{C}) \subset \mathcal{A}$. Si $E \in \mathcal{A}$, E o E^c es numerable, luego E o E^c esta en $\sigma(\mathcal{C})$; es decir, $E \in \sigma(\mathcal{C})$. $\therefore\ \mathcal{A} = \sigma(\mathcal{C})$.

Relación 2

(1) (a) $\lambda E \subset \bigcup_{n=1}^{\infty} I_n \Leftrightarrow E \subset \bigcup_{n=1}^{\infty} \lambda^{-1} I_n$

$$m^*(\lambda E) = \lambda \inf \{\lambda^{-1} \sum_{n=1}^{\infty} \ell(I_n) : E \subset \bigcup_{n=1}^{+\infty} \lambda^{-1} I_n\} = \lambda\, m^*(E).$$

(b) E medible $\Rightarrow \lambda E$ es medible

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c), \quad \forall A \subset \mathbb{R}.$$

$$m^*(A \cap (\lambda E)) + m^*(A \cap (\lambda E)^c)$$

$$= m^*(\lambda ((\lambda^{-1}A) \cap E)) + m^*(\lambda (\lambda^{-1}A) \cap E^c)$$

$$= \lambda [m^*((\lambda^{-1}A) \cap E) + m^*((\lambda^{-1}A) \cap E^c)$$

$$= \lambda m^*(\lambda^{-1}A) = m^*(\lambda \lambda^{-1}A) = m^*(A).$$

(2) E y F medibles. Entonces $m(E)+m(F) = m(E\cup F)+m(E\cap F)$

$E = E\cap F + E\setminus F \Rightarrow m(E) = m(E\cap F) + m(E\setminus F)$

$F = E\cap F + F\setminus E \quad m(F) = m(E\cap F) + m(F\setminus E)$

$m(E)+m(F) = m(E\cap F) + m(E\cap E) + m(E\setminus F) + m(F\setminus E).$

$E\cup F = E\cap F \cup E\setminus F \cup F\setminus E$, unión disjunta. luego

$$m(E\cup F) = m(E\cap F) + m(E\setminus F) + m(F\setminus E)$$

(3) $\{E_n\}_{n=1}^{\infty}$ medibles; $\liminf E_n = \bigcup_{k\in\mathbb{N}} \bigcap_{n>k} E_n.$

- $A_k = \bigcap_{n>k} E_n$; $A_1 \subset A_2 \subset A_3 \cdots\cdots$; $\bigcup_{n=1}^{\infty} A_n = \liminf E_n$

$\therefore m(\liminf E_n) = m(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n) \le \liminf_{n\to+\infty} m(E_n)$, puesto que $m(A_n) \le m(E_n)$, $\forall n\ge 1.$

- $\limsup E_n = \bigcap_k \bigcup_{n>k} E_n$; $m(\bigcup_{n=1}^{\infty} E_n) < +\infty.$ $\bigcap_{n=1}^{\infty} A_n = \limsup E_n$;

$A_k = \bigcup_{n>k} E_n$; $A_1 \supset A_2 \supset \cdots \supset A_k \cdots$,

$m(\limsup E_n) = m(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to+\infty} m(A_n)$,

Como $m(A_1) < +\infty$,

$m(A_n) \ge m(E_n)$ y $\lim_{n\to+\infty} m(A_n) \ge \limsup_{n\to+\infty} m(E_n).$

(4) $A\subset\mathbb{R}$, $B\in\mathcal{M}$.

(a) $\exists\, G = \bigcap_{n=1}^{+\infty} V_n$, V_n abierto, $A\subset G$ y $m(G) = m^*(A)$

. Si $m^*(A) = +\infty$, $G = \mathbb{R}$.

. Si $m^*(A) < +\infty$, $\forall \varepsilon>0$, $\exists \{I_j^{\varepsilon}\}_{j=1}^{\infty}$ intervalos abiertos,

$A \subset \bigcup_{j=1}^{\infty} I_j^{\varepsilon}$, $m(\bigcup_{j=1}^{\infty} I_j^{\varepsilon}) \le \sum_{j=1}^{+\infty} \ell(I_j^{\varepsilon}) < m^*(A) + \varepsilon.$

Sea $V^{\varepsilon} = \bigcup_{j=1}^{\infty} I_j^{\varepsilon}$, $G = \bigcap_{n=1}^{+\infty} V^{\frac{1}{n}}$, $A\subset G$ y

$m^*(A) \le m^*(G) \le m^*(V^{\frac{1}{n}}) \le m^*(A) + \frac{1}{n}$, $\forall n\ge 1$; es decir

$m^*(A) = m^*(G)$

(b) Si $m^*(A) < +\infty$, $B \subset A$ y $m(B) = m^*(A) \Rightarrow A \in \mathcal{M}$.

Como B es medible, $m^*(A) = m^*(A \cap B) + m^*(A \setminus B)$
$= m^*(B) + m^*(A \setminus B) = m^*(A) + m^*(A \setminus B)$ y $m^*(A) < +\infty$;
es decir, $m^*(A \setminus B) = 0$, $A \setminus B \in \mathcal{M}$ y $A = B \cup A \setminus B \in \mathcal{M}$.

(c) Si $m(B) < +\infty$ y $A \subset B$, A es medible $\Leftrightarrow$ $m(B) = m^*(A) + m^*(B \setminus A)$

$(\Rightarrow)$ Sabemos que $m^*(A) \leq m^*(B) < +\infty$ y $m(B) = m^*(B)$
$= m^*(B \cap A) + m^*(B \setminus A) = m^*(A) + m^*(B \setminus A)$;

$(\Leftarrow)$ Si $m(B) = m^*(A) + m^*(B \setminus A)$, por (a) $\exists$ G medible tal que $A \subset G$ y $m(G) = m(A^*)$. Si $E = G \cap B$, E es medible, $A \subset E$, $m(E) = m^*(A)$. Como E es medible, $m^*(A$
$B \setminus E \subset B \setminus A$ y $m(B) = m(E) + m(B \setminus E)$, $\therefore$ $m(B$
$+ m^*(B \setminus A) = m(B) = m(B \cap E) + m(B \setminus E) = m^*(B \setminus A)$. Por (b), $B \setminus A$ es medible y $A = B \setminus (B \setminus A)$ también.

---

(5) $f : \mathbb{R} \to \mathbb{R}$ medible. Entonces $1/f$ es medible.

Suponemos que $f(x) \neq 0$, $\forall x$, en caso contrario no podemos definir $\frac{1}{f}$ a no ser que $E = \{x \in \mathbb{R} : f(x) = 0\}$ tenga medida nula. En el primer caso:

$$\mathbb{R} \longrightarrow \mathbb{R}\setminus\{0\} \qquad \mathbb{R}\setminus\{0\} \xrightarrow{\ g\ } \mathbb{R}$$
$$x \rightsquigarrow f(x) \qquad\qquad u \rightsquigarrow \frac{1}{u}$$

continua

$\frac{1}{f} = g \circ f$. Si V abierto en $\mathbb{R}$

$$\left(\tfrac{1}{f}\right)^{-1}(V) = (g \circ f)^{-1}(V) = f^{-1}(\underbrace{g^{-1}(V)}_{\text{abierto}}) \in \mathcal{M}.$$

Si E tiene medida nula: consideramos $\mathbb{R} \setminus E \longrightarrow \mathbb{R}\setminus\{0\}$,
$x \rightsquigarrow f(x)$

$g: \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$, $\frac{1}{f} = g \circ f$ en $\mathbb{R}\setminus E$. $\therefore$ $\frac{1}{f}$ es medible en $\mathbb{R}$.

g la podemos definir como queramos en E, pues $m(E)=0$. (5

⑥ $h, g : \Omega \longrightarrow \overline{\mathbb{R}}$ medibles; Ω medible.

- $\{x\in\Omega : h(x) < g(x)\} = \bigcup_{r\in\mathbb{Q}} \{x\in\Omega : h(x)<r\} \cap \{x\in\Omega : g(x)>$

$= \bigcup_{r\in\mathbb{Q}} h^{-1}([-\infty, r)) \cap g^{-1}((r,+\infty])$

$\{x\in\Omega : h(x)<g(x)\} \cup \{x\in\Omega :$

- $\{x\in\Omega : h(x)\le g(x)\} =$

$h(x)=g(x)\}$. $h^{-1}(\{+\infty\}) \cap g^{-1}(\{+\infty\}) \cup h^{-1}(\{-\infty\})$

El segundo conjunto es $\qquad \tilde{\Omega} = \{x\in\Omega :$

$\cap\, g^{-1}(\{-\infty\}) \cup \{x\in\tilde{\Omega} : h(x)=g(x)\}$. Aquí $h, g : \tilde{\Omega} \longrightarrow \mathbb{R}$ es medible,

$h(x)\neq\pm\infty$ y $g(x)\neq\pm\infty\}$. En $\tilde{\Omega}$,

$h-g : \tilde{\Omega} \longrightarrow \mathbb{R}$ es medible y

$\{x\in\tilde{\Omega} : h(x)=g(x)\} = (h-g)^{-1}(\{0\})$

con $h-g$ como función de $\tilde{\Omega}$ en $\mathbb{R}$.

- Finalmente $\{x\in\Omega : h(x)=g(x)\} = \{x\in\Omega : h(x)\le g(x)\}\setminus\{x\in\Omega :$

$h(x)<g(x)\}$.

⑦ $E\subset\mathbb{R}$ medible, $f: E\longrightarrow\mathbb{R}$ es cont. p.c.t. $x\in E$. Entonces f es medible.

Si $A = \{x\in E : f$ es cont. en $x\}$, $g = f|A$ es continua en A en $\mathbb{R}$ y por tanto medible. A es medible y $E\setminus A$ tiene medida nula.

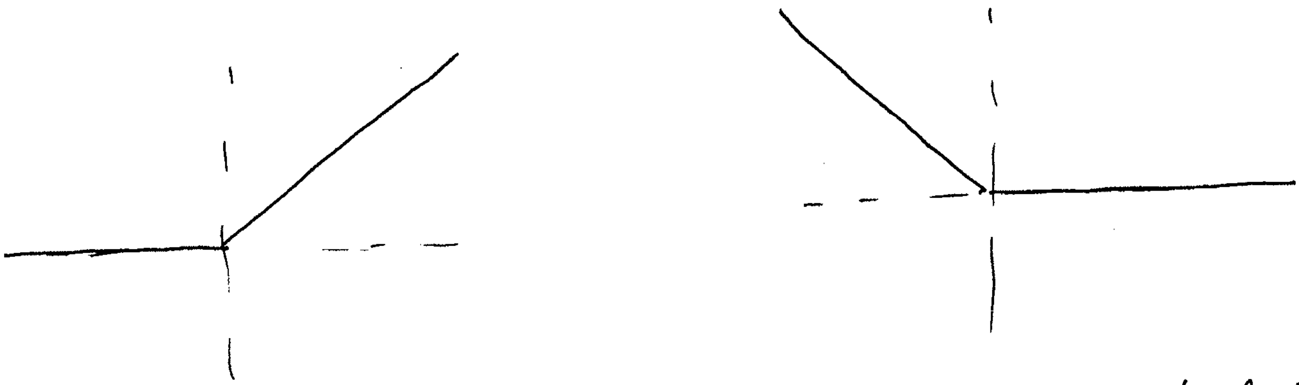
⑧ $f:\mathbb{R}\longrightarrow\mathbb{R}$ es tal que $\exists\, f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, $\forall x\in\mathbb{R}$. $\qquad$ es medible y

Sea $f_n(x) = n\left[f(x+\frac{1}{n}) - f(x)\right]$, $f_n : \mathbb{R}\longrightarrow\mathbb{R}$

$f'(x) = \lim_{n\to+\infty} f_n(x)$. ∴ f' es medible.

(9). $\overline{\mathbb{R}} \to [0,+\infty]$, $x \mapsto \max\{x,0\}$. $\quad\overline{\mathbb{R}} \to [0,+\infty]$, $x \mapsto -\min\{x,0\}$

son continuas.



$f^+ = \max\{f,0\}$, $\quad f^- = -\min\{f,0\}$, composición de medible con continua.

• $f = f^+ - f^-$, $\quad |f| = f^+ + f^-$

$f^+ = \frac{f+|f|}{2}$, $\quad f^- = \frac{|f|-f}{2}$.

• Si f es no medible, f^+ o f^- es no medible.

• $f(x) = \begin{cases} 1, & x\in E \\ -1, & x\notin E \end{cases}$, $\quad E \notin \mathcal{M}$, $\quad |f|\equiv 1$.

Relación 3

(3) $f, g \in L^1(\mathbb{R})$

$|\min(f,g)| \le |f|+|g|$

$\min(f,g) \le f, g \Rightarrow \int \min(f,g) \le \int f$ y $\int g$

∴ $\int \min(f,g) \le \min\left(\int f, \int g\right)$

⑤ $$\int_\varepsilon^1 \frac{\sin(1/x)}{x}\,dx = \left\{\begin{array}{l} x = \frac{1}{u} \\ dx = \frac{du}{u^2} \end{array}\right\} = \int_1^{1/\varepsilon} \frac{u \sin(u)}{u^2} = \int_1^{1/\varepsilon} \frac{\sin u}{u}\,du.$$

$\exists \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{\sin(1/x)}{x}\,dx$, pero $\frac{\sin(1/x)}{x}$ no es integrable Lebesgue en $[0,1]$.

⑥ $f(x) = \left(\int_0^x e^{-t^2} dt\right)^2$, $g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\,dt$.

(a) $f(x) + g(x) = \frac{\pi}{4}$.

$f(x) = \left(x \int_0^1 e^{-x^2t^2} dt\right)^2$, $f'(x) = 2\sqrt{f}\, e^{-x^2}$, si $x>0$.

$$g'(x) = \int_0^1 \frac{-2x(1+t^2)\, e^{-x^2(1+t^2)}}{1+t^2}\,dt = -2x \int_0^1 e^{-x^2t^2} dt\, e^{-x^2}$$

$$= -2\sqrt{f}\, e^{-x^2} = -f'(x), \text{ si } x>0.$$

$\therefore\ f(x) + g(x) = f(0) + g(0) = \int_0^1 \frac{dt}{1+t^2} = \arctan(1) = \frac{\pi}{4}$.

(b) $g(+\infty) = 0$, $f(+\infty) = \left(\int_0^{+\infty} e^{-t^2} dt\right)^2 = \frac{\pi}{4}$, $\int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

⑦ (a) $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$.

Por inducción: $\int_{-\infty}^{+\infty} x^{2n} e^{-x^2} dx = \int_{-\infty}^{+\infty} x^{2n-1} (-2x e^{-x^2}) \frac{dx}{-2}$

$$= -\frac{1}{2}\left[x^{2n-1} e^{-x^2} \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} (2n-1)\, x^{2(n-1)} e^{-x^2} dx. \right.$$

$\therefore\ \int_{-\infty}^{+\infty} x^{2n} e^{-x^2} dx = \frac{2n-1}{2} \int_{\mathbb{R}} x^{2(n-1)} e^{-x^2} dx$

(b) $a \geq 0$

$$\int_{\mathbb{R}} e^{-x^2} \cos(ax)\, dx = \int_{\mathbb{R}} \sum_{n=0}^{\infty} (-1)^n \frac{(ax)^{2n}}{(2n)!} e^{-x^2} dx.$$

$$= \int_{\mathbb{R}} \lim_{n \to +\infty} S_N(x)\, dx, \qquad S_N(x) = \sum_{n=0}^{N} (-1)^n \frac{(ax)^{2n}}{(2n)!} e^{-x^2},$$

$$|S_N(x)| \leq \Big(\sum_{n=0}^{\infty} \frac{(ax)^{2n}}{n!^2}\Big) e^{-x^2} \leq e^{-x^2 + 2a|x|},$$

pues $\dfrac{|ax|^n}{n!} \leq e^{a|x|}$ y $\displaystyle\sum_{n \geq 0} \frac{|ax|^n}{n!} = e^{a|x|}$.

$$\therefore \int_{\mathbb{R}} e^{-x^2} \cos(ax)\, dx = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(-1)^n}{(2n)!} a^{2n} \int_{\mathbb{R}} x^{2n} e^{-x^2} dx$$

$$= \lim_{N \to +\infty} \sum_{n=0}^{N} \frac{(-1)^n}{\cancel{(2n)!}} a^{2n} \frac{\cancel{(2n)!}\sqrt{\pi}}{4^n n!}$$

$$= \lim_{N \to +\infty} \sqrt{\pi} \sum_{n=0}^{N} (-1)^n \frac{a^{2n}}{4^n n!} = \sqrt{\pi}\, e^{-a^2/4}.$$

(c) Si $p > 0$, $\displaystyle\int_0^1 x^{p-1} \frac{1}{x-1} \log x = \int_0^1 x^{p-1} \log\left(\frac{1}{x}\right) \sum_{n=0}^{\infty} x^n dx$

$$= \sum_{n=0}^{+\infty} \int_0^1 x^{n+p-1} \log\left(\frac{1}{x}\right) dx. \cdots$$

(8) $\displaystyle\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt$ es C^{∞} ----

$f(x,t) = e^{-t + (x-1)\log t}$, si $\varepsilon \leq x \leq R$,

$$|f(x,t)| \le e^{-t + |x-1|\,|\log t|} \le e^{-t + \max\{1-\varepsilon, R-1\}|\log t|}.$$

$$\partial_x^\alpha f(x,t) = (\log t)^\alpha e^{-t + (x-1)\log t}$$

$$|\partial_x^\alpha f(x,t)| \le |\log t|^\alpha e^{-t + \max\{1-\varepsilon, R-1\}|\log t|}.$$

(4) $f, f_n : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ integrables, $f, f_n \ge 0$, $f_n(x) \longrightarrow f(x)$ p.c.t. $x \in \Omega$. Entonces, si $\int_\Omega f_n\,dx \to \int_\Omega f\,dx$, se verifica que $\int_\Omega |f_n - f|\,dx \to 0$.

Dem: $0 \le (f-f_n)^+ \le f^+$ y $(f-f_n)^+(x) \to 0$ en c.t. $x \in \Omega$. De Aplicar TCD para concluir que $\int_\Omega (f-f_n)^+ dx \to 0$. forma análoga, $\int_\Omega (f- \downarrow 0$ pa TC

Rel 4

(2) $\ln z = \lim_{n\to+\infty} n(\sqrt[n]{z} - 1)$, $\forall z > 0$

$$n(\sqrt[n]{z}-1) = n(z^{1/n}-1) = n(e^{\frac{1}{n}\log z} - 1)$$

$$= \left(\frac{e^{\frac{1}{n}\log z} - 1}{\frac{1}{n}\log z} \right) \log z \longrightarrow (e^u)'_{u=0} \log z = \log z.$$

(3) $$\int \frac{1}{x^2}\chi_{[0,1]}(x)\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{dx}{x^2} = +\infty \quad \text{(TCM)}$$

$$\int \frac{1}{\sqrt{x}}\chi_{[0,1]}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} = 2. \quad \text{(TCM}$$

(4) $\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,dx.$

$$f_n(x) = \frac{1+nx^2}{(1+x^2)^n}, \quad \lim_{n\to+\infty} f_n(x) = 0.$$

CV

$$\frac{1+nx^2}{(1+x^2)^n} = e^{-n\log(1+x^2)}(1+nx^2)$$

$$= (1+nx^2)\, e^{-nx^2 \frac{\log(1+x^2)}{x^2}} \leq (1+nx^2)\, e^{-nx^2}.$$

$$0 \leq \int_0^1 f_n(x)\,dx \leq \int_0^1 (1+nx^2)\, e^{-nx^2}\,dx = \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} (1+y^2)\, e^{-y^2}\,dy \leq \frac{1}{\sqrt{n}} \int_0^{+\infty} (1+y^2)\, e^{-y^2}\,dy.$$

(5) $\int_0^n \left(1+\frac{x}{n}\right)^n e^{ax}\,dx, \quad a<-1.$

$$\left(1+\frac{x}{n}\right)^n e^{ax} \to e^{(1+a)x}.$$

TCD

$$\left(1+\frac{x}{n}\right)^n e^{ax} = e^{ax+n\log(1+\frac{x}{n})} = e^{ax + x\left[\frac{\log(1+x/n)}{x/n}\right]} \leq e^{(a+1)x}, \text{ en } (0,+$$

(6) (a) $\lim_{n\to\infty} \int_1^n \left(1-\frac{t}{n}\right)^n \log t\,dt = \int_1^\infty e^{-t}\log t\,dt.$

$\left(1-\frac{t}{n}\right)^n \log t \to e^{-t}\log t,$ en $(0,+\infty)$ (11)

$\left(1-\frac{t}{n}\right)^n \log t = e^{n\log\left(1-\frac{t}{n}\right)} \log t$ <u>TCD</u>

$= e^{+t\frac{\log(1-t/n)}{+t/n}}(\log t) \leq e^{-t}\log t,$ en $[1,n]$

pues $\frac{\log(1-x)}{x} \leq -1,$ en $[0,1]$.

(b) $\lim_{n\to\infty}\int_0^1 \left(1-\frac{t}{n}\right)^n \log t\,dt = \int_0^1 e^{-t}\log t\,dt$

Por TCD pues $\frac{\log(1-x)}{x}$ está acotada en $[0$

y $\left| e^{t\frac{\log(1-t/n)}{t/n}} \log t\right| \leq e^{Mt}|\log t| \leq e^{M}\log 1/t$.

<u>TCD</u>

⑦ $A = \lim_{n\to\infty}\int_a^{+\infty}\frac{n^2 x e^{-n^2x^2}}{1+x^2}\,dx.$

- $A=0,$ si $a>0$.

$f_n(x) \to 0$ en $(a,+\infty)$.

$0 \leq f_n(x) = \frac{n^2x^2e^{-n^2x^2}}{x(1+x^2)} \leq \frac{M}{x(1+x^2)} \in L^1(a,+$

<u>TCD</u>

- $A > \frac{1}{4}$, si $a=0$.

$$\int_0^\infty \frac{n^2 x\, e^{-n^2x^2}}{1+x^2}\,dx. = \int_0^\infty \frac{y e^{-y^2}}{1+ y^2/n^2}\,dy \xrightarrow[\text{TCD}]{}$$

$$\int_0^\infty y e^{-y^2} dy .$$

(8) $f \geq 0$, $0 < \int f < \infty$, $\alpha > 0$.

$$\int n \log\left[1+ \left(\frac{f(x)}{n}\right)^\alpha\right] dx.$$

$$\geq \int_{f\geq \delta} n \log\left(1+ \frac{\delta^\alpha}{n^\alpha}\right) \geq |\{x: f(x) \geq \delta\}|\, n \log\left(1+ \frac{\delta^\alpha}{n^\alpha}\right)$$

$\to +\infty$, si $0<\alpha<1$.

$$n \log\left(1+ \frac{f(x)}{n}\right) = \frac{\log\left(1+ \frac{f(x)}{n}\right)}{\frac{f(x)}{n}}\, f(x) \to f(x).$$

$\frac{\log(1+u)}{u}$ está acotada en $[0,+\infty)$. $\underline{\underline{\text{TCD}}}$

Si $1<\alpha<+\infty$,

$$n \log\left(1+ \frac{f(x)^\alpha}{n^\alpha}\right) \to 0 \quad d$$

$$\frac{n \log\left(1+ \frac{f(x)^\alpha}{n^\alpha}\right)}{\frac{f(x)^\alpha}{n^\alpha}} \left(\frac{f(x)}{n}\right)^\alpha .$$

$\leq n \left(\frac{f|x|}{u}\right)^{\alpha} M \leq n \frac{f|x|}{u} M \leq M f|x|$ si $f|x|$

(13)

Si $f|x| \geq u$

$n \lg \left(1 + \frac{f|x|^{\alpha}}{u^{\alpha}}\right) \leq f(x) \lg \frac{\left(1 + \left(\frac{f}{u}\right)^{\alpha}\right)}{f/u} \leq M f$

pues $\frac{\lg(1+u^{\alpha})}{u} \leq M$ en $[1, +\infty)$.

y <u>TCD</u>

(9) $\int_0^1 \log^2 x \left(\sum_{n=1}^{\infty} n x^{u-1}\right) dx = \sum_{n=1}^{\infty} \int_0^1 n x^{u-1} \log^2 x\, dx.$ <u>TCM</u>

$\int_0^1 n x^{u-1} \lg^2 x\, dx = x^u \lg^2 x \Big]_0^1 - \int_0^1 2 x^{u-1} \lg x$ - - - -

(10) $\int_0^{\infty} \chi_{(0,n)} \left(1 - \frac{x}{2u}\right)^n e^x dx.$

$\left(1 - \frac{x}{2u}\right)^n e^x = e^{x + n \lg\left(1 - \frac{x}{2u}\right)}$

$= e^{x + \left[\frac{\lg(1 - x/2u)}{-\frac{x}{2u}}\right]\left(\frac{-x}{2}\right)} \longrightarrow e^{\frac{x}{2}}.$

$e^{x + \frac{\lg\left(1 - \frac{x}{2u}\right)}{x/2u}\left(\frac{x}{2}\right)} \geq e^{+x/2}$, en $(0, u$

pues $x + \lg(1-x)$ es decreciente en $(0,1)$

y $\frac{\log(1-x)}{x} \geq -1$, en $(0,1)$.

(14)

$$\therefore \int_0^{+\infty} \chi_{(0,n)} \left(1 - \frac{x}{2n}\right)^n e^{-x}\, dx \quad (TCM)$$

$$\geq \int_0^n e^{x/2} dx \longrightarrow +\infty, \quad \text{si} \quad n \to +\infty.$$

(11) (a) $\displaystyle\int_0^{+\infty} \frac{\sin(x/n)}{\left(1+\frac{x}{n}\right)^n}\, dx.$

$$\lim_{n\to\infty} \frac{\sin(x/n)}{\left(1+\frac{x}{n}\right)^n} = \frac{0}{e^x} = 0, \quad \text{si} \quad x > 0.$$

También:

$$\int_0^{+\infty} \frac{n \sin x}{(1+x)^n}\, dx \ ; \quad \frac{n \sin x}{(1+x)^n} \longrightarrow 0, \quad \text{si } n \to +\infty$$

y

$$\left|\frac{n \sin x}{(1+x)^n}\right| = \frac{n|\sin x|}{1+\binom{n}{1}x + \binom{n}{2}x^2 + \cdots}$$

$$\leq \frac{n|\sin x|}{1 + nx + \underbrace{n(n-1)x^2 + \cdots}_{\geq 0}}$$

$$\leq \begin{cases} \dfrac{|\sin x|}{x} \leq 1, & \text{si } x > 0. \\[2ex] \dfrac{n|\sin x|}{n(n-1)x^2} \lesssim \dfrac{1}{x^2}, & \text{si } x > 0, \end{cases} \quad \leq \chi_{(0,1)} + \frac{1}{x^2}\chi_{(1,\infty)}.$$

$$\in L^1(0,+\infty)$$

TCD

(b) $\displaystyle\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx.$

$$\frac{n\sin(x/n)}{x(1+x^2)} = \frac{\sin(x/n)}{x/n}\;\frac{1}{1+x^2} \longrightarrow \frac{1}{1+x^2},$$

si $n \to +\infty$, $\quad x>0.$

$$\left|\frac{n\sin(x/n)}{x(1+x^2)}\right| \le \frac{1}{1+x^2};$$

$\underline{\underline{TCD}}$

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} < +\infty$$

(c) $$\int_a^{+\infty} \frac{n\,dx}{1+n^2x^2} = \left\{\begin{array}{l} nx=y \\ dx=\frac{1}{n}dy \end{array}\right\} = \int_{an}^{+\infty} \frac{dy}{1+y^2}.$$

$a>0$, fa TCD $\longrightarrow 0.$

$a=0$, fácil

$a<0$, TCM o calcular $\longrightarrow \displaystyle\int_{\mathbb{R}} \frac{dy}{1+y^2} = \pi.$

(12) $\displaystyle\int_0^n \left(1-\frac{x}{n}\right)^n e^{-x}\,dx.$

$$\left(1-\frac{x}{n}\right)^n e^{-x}\chi_{(0,n)}(x) =$$

$$= \chi_{(0,n)}(x)\, e^{-x+n\lg\left(1-\frac{x}{n}\right)}$$

$$= \chi_{(0,n)}(x)\, e^{-x+\frac{\lg\left(1-\frac{x}{n}\right)}{\left(-\frac{x}{n}\right)}(-x)} \longrightarrow e^{-2x}, \text{ en } (0,+\infty).$$

Además,

$$\left|\chi_{(0,n)}(x)\, e^{-x+n\log(1-x)}\right| \le e^{-x}, \text{ en } (0,+\infty)$$

y **TCD**. El límite es

$$\int_0^{+\infty} e^{-2x}\,dx.$$

13 $$\int_0^{+\infty} \frac{1}{\left(1+\frac{x}{n}\right)^n x^{1/n}}\,dx.$$

$$\frac{1}{\left(1+\frac{x}{n}\right)^n x^{\frac{1}{n}}} = x^{-\frac{1}{n}} e^{-n\lg\left(1+\frac{x}{n}\right)}$$

$$= x^{-\frac{1}{n}} e^{-\left(\frac{\lg\left(1+\frac{x}{n}\right)}{\frac{x}{n}}\right)x} \longrightarrow x^0 e^{-x} = e^{-x}, \text{ en } (0,+\infty)$$

Ahora, si $0 \le x \le 1$,

$$0 \le \frac{1}{\left(1+\frac{x}{n}\right)^n x^{1/n}} \le \frac{1}{x^{\frac{1}{n}}} \le \frac{1}{x^{1/2}},$$ si $n \ge 2$;

Si $x \ge 1$,

$$0 \le \frac{1}{\left(1+\frac{x}{n}\right)^n x^{\frac{1}{n}}} \le \frac{1}{\left(1+\frac{x}{n}\right)^n} \le \frac{1}{\binom{n}{2}\left(\frac{x}{n}\right)^2}$$

$$= \frac{n^2}{\frac{n!}{2!(n-2)!} x^2} = \frac{2n^2}{n(n-2)x^2} \leq \frac{1}{x^2}.$$

$$\therefore \quad 0 \leq \frac{1}{x^{\frac{1}{n}}\left(1+\frac{x}{n}\right)^n} \leq \frac{1}{\sqrt{x}} \chi_{(0,1)} + \frac{1}{x^2} \chi_{(1,+\infty)},$$

si $n \geq 2$. Aplicar TCD para obtener que el límite es $\int_0^\infty e^{-x}dx = -e^{-x}\Big]_0^{+\infty} = 1.$

(14) $\displaystyle\int_0^1 \frac{nx \sin x}{1+(nx)^p}dx, \quad 1<p<2.$

$$\frac{nx \sin x}{1+(nx)^p} \to 0, \quad \text{si } n \to +\infty, \quad \forall x \in (0,1)$$

$$\left|\frac{nx \sin x}{(1+nx)^p}\right| \leq n^{1-p} x^{1-p} |\sin x|$$

$$\leq x^{2-p} \text{ en } (0,1)$$

que es integrable en $(0,1)$, pues $2-p>0$.

(15) $\displaystyle\int_0^1 \frac{\log(n+x)}{n} e^{-x} \cos x \, dx.$

$$\frac{\log(n+x)}{n} e^{-x} \cos x \to 0, \quad \text{si } n \to \infty, \quad x \in (0,1)$$

$$\left|\frac{\lg(n+x)}{n} e^{-x} \cos x\right| \leq \frac{\lg(n+1)}{n} \to 0, \text{ si } n \to +\infty$$

① $f, f_n : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ integrables, $f_n(x) \to f(x)$ (18)

a c.t. $x \in \Omega$. Probar que

$$\int_\Omega |f_n - f|\,dx \longrightarrow 0 \iff \int |f_n|\,dx \to \int_\Omega |f|\,dx.$$

$\Rightarrow$) (Trivial)

$\Leftarrow$) Sea $g_n = |f_n| - |f_n - f|$; , $|g_n| \leq \big||f_n| - |f_n - f|\big|$

$\leq |f_n - (f_n - f)| = |f|$, $g_n(x) \to |f(x)|$; en casi todo $x \in \Omega$

por TCD:

$$\int_\Omega g_n\,dx \longrightarrow \int_\Omega |f|\,dx$$

pero

$$\int_\Omega g_n\,dx = \int_\Omega |f_n|\,dx - \int_\Omega |f_n - f|\,dx \quad y$$

también $\int_\Omega |f_n|\,dx \longrightarrow \int_\Omega |f|\,dx$ por hipótesis; es decir,

$\int_\Omega |f_n - f|\,dx \longrightarrow 0$, si $n \longrightarrow +\infty$.

Rel 5

① $f(x,y) = \begin{cases} 1, & x \in \mathbb{Q}\cap[0,1],\ y\in[0,1], \\ 0, & \ldots \end{cases}$ $\qquad \int_{[0,1]\times[0,1]} f\,dm_2 = 0.$

$f = \chi_{\mathbb{Q}\times[0,1]}$, $\quad \mathbb{Q}\times[0,1] \in \mathcal{M}_1\times\mathcal{M}_1 \subsetneq \mathcal{M}_2.$

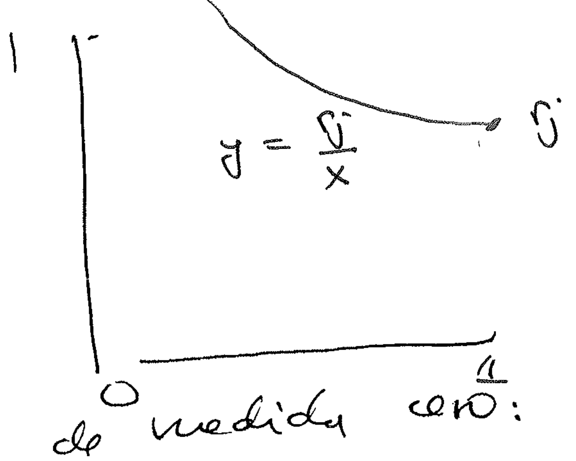
⑦ $f(x,y)=\begin{cases}1, & \text{si } x\cdot y\in\mathbb{Q} \\ 0, & \ldots\end{cases}$, en $[0,1]\times[0,1]$.

$\mathbb{Q}\cap[0,1] = \{r_1, r_2, r_3, \ldots\}$ numerable. Define

$$f_n(x,y) = \begin{cases} 1, & \text{si } x\cdot y \in \{r_1,r_2,\ldots,r_n\}. \\ 0, & \text{en otro caso.}\end{cases}$$

f_n es medible, $\lim_{n\to+\infty} f_n(x,y) = f$, en $[0,1]\times[0,1]$, $0\le f_1\le f_2 \le \cdots \le f_n \le \cdots \le f$,

f_n es simple con valores no nulos en unión de gráficas finitas sobre $[0,1]$.



Que es un conjunto de medida cero: la unión finita en $\mathbb{R}^2$.

de curvas $y = \frac{r_1}{x}, y = \frac{r_2}{x}, \dots, y = \frac{r_n}{x}$ dentro de $[0,1]^2$
$[0,1]$. $\therefore$

$$\int_{[0,1]\times[0,1]} f_n \, dx\, dy = 0 \quad y \quad \underline{\text{TCM}}. \text{ implica } \int_{[0,1]^2} f \, dm_2 = 0.$$

(8) $E \subset [0,1]\times[0,1]$ medible, $m_1(E^x) \leq 1/2$,
$E^x = \{y \in [0,1] : (x,y) \in E\}$, p.c.t. x en $[0,1]$.
Entonces $m_1(\{y \in [0,1] : m_1(E_y) = 1\}) \leq 1/2$.

$$|E| = \int_0^1 m_1(E^x)\,dx \leq \frac{1}{2}.$$

$$\| $$

$$\int_0^1 m_1(E_y)\,dy \geq \int_{\{y \in [0,1] :\, m_1(E_y) = 1\}} dy$$

$$= m_1(\{y \in [0,1] : m_1(E_y) = 1\}).$$

(3)
$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{(x^2+y^2)^2}, & \text{si } (x,y) \neq (0,0), \\ 0, & \text{si } (x,y) = (0,0). \end{cases}$$

$$\int_0^1 f(x,y)\,dy = \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy = \left\{ \begin{array}{l} y = xu \\ dy = x\,du \end{array} \right\}$$

$$= \frac{1}{x}\int_0^{1/x} \frac{1-u^2}{(1+u^2)^2}\,du = \frac{1}{x}\left[\int_0^1 \frac{1-u^2}{(1+u^2)^2}\,du + \int_1^{1/x} \frac{1-u^2}{(1+u^2)^2}\,du\right]$$

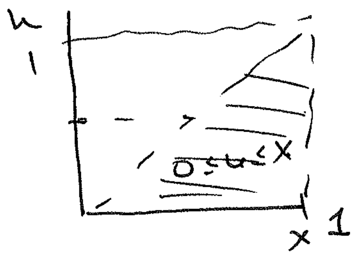
$$\int_1^{1/x} \frac{1-u^2}{(1+u^2)^2}\,du = \left\{\begin{array}{l} v = 1/u \\ du = \frac{dv}{v^2} \end{array}\right\} = -\int_x^1 \frac{1-u^2}{(1+u^2)^2}\,du.$$

$$\therefore \int_0^1 f(x,y)\,dy = \frac{1}{x}\int_0^x \frac{1-u^2}{(1+u^2)^2}\,du.$$

$$\int_0^1 f(x,y)\,dx = \int_0^1 \frac{x^2-y^2}{(x^2+y^2)}\,dx = \left\{\begin{array}{l} x = yu \\ dx = y\,du \end{array}\right\}.$$

$$= \frac{1}{y}\int_0^{\frac{1}{y}} \frac{u^2-1}{(u^2+1)^2}\,du = -\frac{1}{y}\int_0^y \frac{1-u^2}{(1+u^2)^2}\,du$$

$$\therefore \int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx = \int_0^1 \frac{1}{x}\left(\int_0^x \underbrace{\frac{1-u^2}{(1+u^2)^2}}_{\geq 0}\,du\right)dx$$



$$= \int_0^1\int_0^x \frac{1}{x}\,\frac{1-u^2}{(1+u^2)^2}\,du\,dx$$

$$= \int_0^1\left(\int_u^1 \frac{1}{x}\,\frac{1-u^2}{(1+u^2)^2}\,dx\right)du = \int_0^1 \log\left(\frac{1}{u}\right)\frac{1-u^2}{(1+u^2)^2}\,du.$$

$$\int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy = -\int_0^1\left(\frac{1}{y}\int_0^y \underbrace{\frac{1-u^2}{(1+u^2)^2}}_{\geq 0}\,du\right)dy$$

$$= - \int_0^1 \log\left(\frac{1}{u}\right) \frac{1-u^2}{(1+u^2)^2}\, du.$$

Como $\displaystyle\int_0^1 \log\left(\frac{1}{u}\right) \frac{1-u^2}{(1+u^2)^2}\, du > 0$, pues

$\log\left(\frac{1}{u}\right) > 0$, en $(0,1]$ y la integral es

finita, pues

$$0 \leq \log\left(\frac{1}{u}\right) \frac{1-u^2}{(1+u^2)^2} \leq \log\left(\frac{1}{u}\right) \text{ en } (0,1]$$

vemos que

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx\right) dy \neq \int_0^1 \left(\int_0^1 f(y,x)\, dx\right) dy.$$

Claramente, f no puede ser integrable en $[0,1]^2$ por Tma. Tonelli.

(4)

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2}, & \text{si } (x,y) \neq (0,0), \\ 0, & \text{si } (x,y) = (0,0). \end{cases}$$

$$\int_{B_{\frac{1}{2}}} |f(x,y)|\, dx\, dy = \frac{1}{2} \int_{(-\pi,\pi)\times(0,\frac{1}{2})} \frac{r^2 |\sin(2\theta)|}{r^4}\, r\, dr\, d\theta$$

$$= \frac{1}{2} \int_{(-\pi,\pi)\times(0,\frac{1}{2})} \frac{1}{r} |\sin(2\theta)| \, dr\, d\theta$$

$$= \frac{1}{2} \underbrace{\left(\int_0^{1/2} \frac{dr}{r} \right)}_{\substack{\parallel \\ +\infty.}} \left(\int_{-\pi}^{\pi} |\sin(2\theta)| \, d\theta \right)$$

$\therefore$ f no es integrable.

$$\int_{-1}^{1} f(x,y)\,dy = \int_{-1}^{1} \frac{xy}{(x^2+y^2)^2}\,dy = 0, \quad \text{si } x \neq 0$$

pues $y \rightsquigarrow f(x,y)$ es impar.

$$\therefore \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)\,dy \right) dx = \int_{-1}^{1} 0\,dx = 0.$$

De forma análoga:

$$\int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)\,dx \right) dy = \int_{-1}^{1} 0\,dy = 0.$$

(2) $$\int_{(0,+\infty)\times\mathbb{R}} \frac{|\sin x|}{x^{3/2}(1+x^2+y^2)}\,dm_2$$

$$\leq \int_{(0,1)\times\mathbb{R}} \frac{dm_2}{\sqrt{x}\,(1+y^2)}$$

$$+ \int_{(1,+\infty)\times\mathbb{R}} \frac{dm_2}{x^{3/2}\,(1+y^2)}$$

$$= \left(\int_0^1 \frac{dx}{\sqrt{x}}\right)\left(\int_{\mathbb{R}} \frac{dy}{1+y^2}\right)$$

$$+ \left(\int_1^\infty \frac{dx}{x^{3/2}}\right)\left(\int_{\mathbb{R}} \frac{dy}{1+y^2}\right) < +\infty.$$

5 $$f(x,y) = \begin{cases} (x-\frac{1}{2})^{-3}, & \text{si } 0<y<|x-\frac{1}{2}|, \\ 0, & \text{resto.} \end{cases}$$

$$\int_{[0,1]^2} |f(x,y)|\,dm_2 = \int_0^1 \left(\int_0^{|x-1/2|} |x-\tfrac{1}{2}|^{-3}\,dy\right) dx.$$

$$= \int_0^1 |x-\tfrac{1}{2}|^{-2}\,dx = \int_0^{1/2} \frac{dx}{(\frac{1}{2}-x)} + \int_{1/2}^1 \frac{dx}{(x-\frac{1}{2})}$$

$$= 2\int_0^{1/2} \frac{du}{u} = +\infty.$$

y pa Tma. Fubini, f no es integrable en $[0,1]^2$?

(6) $$\int_{[0,1]\times[0,+\infty]} |e^{-y}\sin(2xy)|\,dm_2$$

$$\leq \int_{[0,1]\times[0,+\infty]} e^{-y}\,dm_2 = \left(\int_0^1 dx\right)\left(\int_0^\infty e^{-y}dy\right) < +\infty.$$

$$\therefore \int_{[0,1]\times[0,+\infty)} e^{-xy}\sin(2xy)\,dm_2 = \int_0^1\left(\int_0^\infty e^{-y}\sin(2xy)\,dy\right)dx$$

$$= \int_0^\infty\left(\int_0^1 e^{-y}\sin(2xy)\,dx\right)dy$$

$$= \int_0^\infty e^{-y}\left[-\frac{1}{2y}\cos(2xy)\right]_0^1 dy$$

$$= \int_0^\infty e^{-y}\frac{1}{2y}(1-\cos(2y))\,dy = \int_0^{+\infty} e^{-y}\frac{\sin^2 y}{y}\,dy,$$

pues $\frac{1}{2}(1-\cos(2y)) = \sin^2 y$.

La integral $\int_0^{+\infty} e^{-y}\sin(2xy)\,dy =$

$$= \int_0^{\infty} (-e^{-y})' \sin(2xy)\,dy = - \int_0^{+\infty} (-e^{-y}) (\sin(2xy))'\,dy$$

26

$$= 2x \int_0^{\infty} e^{-y} \cos(2xy)\,dy$$

$$= 2x \int_0^{+\infty} (-e^{-y})' \cos(2xy)\,dy$$

$$= 2x \left\{ \left. -e^{-y} \cos(2xy) \right]_0^{+\infty} - \int_0^{\infty} (-e^{-y}) (-2x \sin 2xy)\,dy \right.$$

$$\left. \right\} = 2x \left(1 - 2x \int_0^{\infty} e^{-y} \sin(2xy)\,dy \right)$$

$\therefore$

$$(1+4x^2) \int_0^{+\infty} e^{-y} \sin(2xy)\,dy = 2x$$

$$\int_0^{+\infty} e^{-y} \sin(2xy)\,dx = \frac{2x}{1+4x^2}.$$

Pa Tma. Tonelli:

$$\int_0^{+\infty} \frac{\sin^2 y}{y} e^{-y}\,dy = \int_0^1 \frac{2x}{1+4x^2}\,dx.$$

$$= \frac{1}{4}\int_0^1 \frac{8x}{1+4x^2}\,dx.$$

$$= \frac{1}{4}\log(1+4x^2)\Big]_0^1 = \frac{\log 5}{4}.$$

Rel 6

(1) Si $1 \le p \le q < \infty$, $\ell^p \subset \ell^q$.

Si $\sum |x_n|^p < +\infty$, entonces $\{x_n\}$ es acotada y $|x_n|^q = |x_n|^{q-p}|x_n|^p$

$$\le \left(\sup_{n\ge 1}|x_n|\right)^{q-p}|x_n|^p.$$

$$\therefore \sum_{n=1}^{+\infty}|x_n|^q \le \left(\sup_{n\ge 1}|x_n|\right)^{q-p}\sum_{n\ge 1}|x_n|^p.$$

(2) $\varphi(x) = \log x$ es cóncava. y

$$\log\left(\int_X f\right) \geq \int_X \log f.$$

$1 \leq f \cdot g, \quad 0 \leq \log f + \log g,$

$$0 \leq \int_X \log f + \int_X \log g \leq \log\left(\int_X f\right) + \log\left(\int_X g\right)$$

$$= \log\left(\left(\int_X f\right)\left(\int_X g\right)\right).$$

$$\therefore \quad 1 \leq \left(\int_X f\right)\left(\int_X g\right)$$

(12) $0 < r < s < \infty, \quad f \in L^r \cap L^s.$

a) $\phi(p) = \log \int f^p$ es convexa en $[r, s]$.

Si $r \leq p_1 \leq p \leq p_2 \leq s, \quad p = \theta p_1 + (1-\theta) p_2.$

$$\int f^p \leq \left(\int f^{p_1}\right)^{\theta} \left(\int f^{p_2}\right)^{1-\theta} \longleftarrow$$

Hölder con exponentes $\frac{1}{\theta}$ y $\frac{1}{1-\theta}$: $\frac{1}{1/\theta} + \frac{1}{1/1-\theta} = 1.$

y como $\log x$ es $\uparrow$ ----

$$\log \int f^p \leq \log\left[\; --- \; \right] \leq \theta \log \int f^{p_1} + (1-\theta) \log \int f^{p_2}$$

b) $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\} = X$

$p = \theta r + (1-\theta)s$

$\log \|f\|_p^p \leq \theta \log \|f\|_r^r + (1-\theta)\log \|f\|_s^s$

$\leq \theta r \log X + (1-\theta) s \log X$

$= p \log X, \quad \therefore p \log \|f\|_p \leq p \log X,$

$\|f\|_p \leq X.$

(13) $1 + t^p \geq (1+t)^p, \quad \forall t > 0, \quad 0 < p < 1$, puesto que

$p\,t^{p-1} - p(1+t)^{p-1} \geq 0$, en $(0,+\infty)$ y

$0 = 1 + 0^p - (1+0)^p \leq 1 + t^p - (1+t)^p$

$\therefore (f(x) + g(x))^p \leq f(x)^p + g(x)^p$, si $f, g \geq 0$

$$\left(\int (f+g)^p\right)^{1/p} \leq \left(\int f^p + \int g^p\right)^{1/p}$$

$$\leq 2^{1/p-1}\left[\left(\int f^p\right)^{1/p} + \left(\int g^p\right)^{1/p}\right],$$

puesto que si $q \geq 1$, $(a+b)^q \leq 2^{q-1}(a^q + b^q)$.