STATIONARY AND SELF SIMILAR SOLUTIONS FOR COAGULATION AND FRAGMENTATION EQUATIONS

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1. Coagulation fragmentation equation.

$$\frac{\partial f}{\partial t}(t,y) = \varepsilon_1 Q(f) + \varepsilon_2 L(f)$$

$$\varepsilon_i \in \{0,1\}, \quad i = 1,2.$$

(i) Coagulation integral.

$$\begin{aligned} Q(f)(y) =& \frac{1}{2}\int_0^y a(y-y',y')f(y-y')f(y')dy' \\ & - \int_0^\infty a(y,y')f(y)f(y')dy' \end{aligned}$$

$$a(y,y') = y^\alpha y'^\beta + y^\beta y'^\alpha,$$

$$-1 \le \alpha \le \beta \le 1, \quad \lambda = \alpha + \beta \in [0,1).$$

(ii) Fragmentation integral.

$$\begin{aligned} L(f)(y) = & \int_y^\infty b(y'',y)f(y'')dy'' \\ & - f(y)\int_0^\infty \frac{y'}{y} b(y,y')dy' \end{aligned}$$

$$b(y, y) = y^\gamma B(\frac{y'}{y}), \quad \gamma \geq -1$$

$$B \geq 0, \mathrm{supp} B \subset [0,1], \int_0^1 y dB(y) < +\infty.$$

If $B(z) = B(1-z)$: Binary fragmentation.

Under these hypothesis no gelling solutions, the solutions of the equation preserve the mass:

$$\int_0^\infty y f(t, y) dy = \int_0^\infty y f(0, y) dy, \quad \text{for all } t > 0.$$

2. Stationary and self similar solutions.

(i) Stationary solutions when $\varepsilon_1 \varepsilon_2 > 0$.

We look for solutions to:

$$Q(f) + L(f) = 0.$$

For binary fragmentation, the equilibriums:

$$a(y, y') M(y) M(y') = b(y + y', y) M(y + y')$$

are a particular case. Not always exist.

(ii) Self similar solutions when $\varepsilon_1 \varepsilon_2 = 0$.

When $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$ no stationary solutions. Other particular solutions: self similar.

When $\varepsilon_2 = 0$: The coagulation equation. Explicit examples:

- For $a(y, y') = 1$, $f(y, t) = 4t^{-2} e^{-\frac{2x}{t}}, \quad t > 0$.
- For $a(y, y') = y + y'$,

$$f(t, y) = (2\pi)^{-1/2} e^{-t} x^{-3/2} e^{-e^{-2t} x/2}.$$

In general, when $\lambda \in [0, 1)$, if f is a solution,

$$f_\mu(t, y) = \mu^{\frac{2}{1-\lambda}} f(\mu t, \mu^{\frac{1}{1-\lambda}} y) \quad \forall \mu > 0$$

is also a solution AND

$$\int_0^\infty f_\mu(t, y) dy = \int_0^\infty f(t, y) dy, \quad \forall \mu > 0.$$

Moreover, if we look for $f \equiv f_\mu$ we obtain:

$$f(t, y) = t^{-\frac{2}{1-\lambda}} g\left(t^{-\frac{1}{1-\lambda}} y\right).$$

$$2g + z\frac{\partial g}{\partial z} + (1 - \lambda) Q(g) = 0.$$

For the fragmentation equation. The self similar solutions have the form:

$$f(t,y) = t^{\frac{2}{1+\gamma}} g(t^{\frac{1}{1+\gamma}} y),$$

$$2g + z\frac{\partial g}{\partial z} - (1+\gamma)L(f) = 0.$$

These particular solutions: may describe the long time behaviour of the solutions to the Cauchy Problem for some families of initial data.

Recent previous results by G. Menon and R. L. Pego (Preprints 2003):

(A) New families of self similar solutions for the kernel $a(y,y') = 1$. These new solutions have no finite mass.

(B) Rederive those already known for the kernels $a(y,y') = 1$ and $a(y,y') = y + y'$.

(C) Determine the domain of atttraction of these solutions. For some initial data with suitably diverging $\lambda+1$ moment, the solution does not converge to any self similar solution.

• We obtain for any $\rho > 0$:

(i) a stationary solution of mass ρ of the coagulation fragmentation equation ,

(ii) self similar solutions of mass ρ of the coagulation and the fragmentation equations.

• Under weaker assumptions on the fragmentation kernel, the stationary solutions of the coagulation fragmentations are measures.

• Uniqueness result only for the fragmentation equation. The difficulty: our uniqueness result for the coagulation fragmentation equation requires some regularity of the solutions.

• Asymptotic behaviour of solutions with finite mass: only for the fragmentation equation. The difficulty: uniqueness of the possible limit.

• Method: A priori estimates, fixed point theorem and compactness.

• We only consider $\lambda \in [0, 1)$. The case $\lambda = 1$ has a different kind of scaling (example above).

3. New results.

• Stationary solutions for coagulation fragmentation equations.

Theorem 1. Assume $B \in L^\infty(0,1)$ and $\alpha \le 0$ or $\gamma = -1$. Then for each $\rho > 0$ there exists at least one solution

$$f \in \bigcap_{k\ge 1} \left(L^1_k \cap L^{k+1}\right), \quad f \in L^1_{-r},\ \forall r \in [0,1),$$

of the stationary coagulation fragmentation equation $Q(f) + L(f) = 0$ such that

$$\int_0^\infty y f(t,y) dy = \rho, \quad \text{forall } t > 0.$$

Moreover, if $\alpha < 0$, this solution satisfies:

$$f \in L^1_{-k}, \quad \text{for all } \ k > 0,$$

We denote, for $r \in \mathbb{R}$

$$f \in L^1_r \Longleftrightarrow M_r = \int_0^\infty y^r\, f(y) dy < +\infty.$$

$$f \in \mathcal{M}_r \Longleftrightarrow M_r = \int_0^\infty y^r\, df(y) < +\infty.$$

$$f \in BV_1 \Longleftrightarrow f \in L^1_{loc},\ \partial_y f \in \mathcal{M}_1$$

• Self similar solutions for coagulation equations.

Theorem 2. For each $\rho > 0$ there exists one self similar solution

$$f = t^{-\frac{2}{1-\lambda}} g\left(y\, t^{-\frac{1}{1-\lambda}}\right), \quad g \in \bigcap_{k \geq 2-\beta} L^1_k$$

of the coagulation equation $\partial_t f = Q(f)$ such that

$$\int_0^\infty y g(y) dy = \rho, \quad \text{forall } t > 0.$$

Moreover, if $\alpha < 0$ then,

$$g \in \bigcap_{k \geq 2-\beta} \left(L^1_{-k} \cap L^1_k\right), \quad \& \quad y^m \partial_y^k g \in L^\infty(\mathbb{R}^+),$$

for all $k \in \mathbb{N}$ and all $m \in \mathbb{Z}$. If $\alpha > 0$, then

$$y^2 f \in L^q(\mathbb{R}^+), \quad \text{for all } \; q \in [1, \frac{1}{\alpha}).$$

• Self similar solutions for fragmentation equations.

Theorem 3. Assume

$$B \in BV_1(0,1) \cap \mathcal{M}_m, \quad m \leq \gamma + 1.$$

Then for each $\rho > 0$ there exists a unique self similar solution

$$f(t,y) = t^{\frac{2}{1+\gamma}} g(t^{\frac{1}{1+\gamma}} y),$$

$$g \in \bigcap_{k \geq m} L^1_k, \quad g \in BV_1(\mathbb{R}^+),$$

of the fragmentation equation $\partial_t f = L(f)$ such that

$$\int_0^\infty y g(y) dy = \rho, \quad \text{forall } t > 0.$$

Moreover, if $\alpha < 0$, this solution satisfies:

$$g \in L^1_{-k}, \quad \text{for all } \ k > 0$$

• Asymptotic behaviour of the solutions to the fragmentation equation.

Theorem 4. Assume

$$B \in BV_1(0,1) \cap M^1_m, \quad m \leq \gamma + 1$$

and

$$f_{in} \in BV_1 \cap L^1_1 \cap L^1_{-1}.$$

Then, there exists a unique solution

$$f \in C\left([0,+\infty); L^1_1 \cap BV_1\right) \cap L^1(0,T; L^1_{\gamma+2}),$$

to the fragmentation equation. Moreover:

$$\lim_{t \to +\infty} \int_0^\infty y|f(t,y) - t^{\frac{2}{1+\gamma}} g(t^{\frac{1}{1+\gamma}} y)|dy = 0.$$

Remark. In the fragmentation case:

$$yf(t,y) \to M\delta, \quad \text{as } t \to +\infty.$$

The limit above: this asymptotic delta-formation is like for the approximation of the identity:

$$\left(t^{\frac{2}{1+\gamma}} y\, g(t^{\frac{1}{1+\gamma}} y)\right)_{t>1}.$$

4. Ingredients of the proofs.

• Fixed point theorem:

Assume: Y Banach space,

$(S_t)_{t\geq 0}$ $Y \to Y$, weakly (sequentially) continuous for any $t > 0$.

$\mathcal{Z}$ a nonempty convex and weakly (sequentially) compact subset of Y

$S_t : \mathcal{Z} \to \mathcal{Z}$ for any $t \geq 0$.

Then, there exists $z_0 \in \mathcal{Z}$ such that $S_t z_0 = z_0$ for any $t \geq 0$.

• Apriori estimates. Consider: $\dfrac{\partial f}{\partial t} = Q(f) + L(f)$

Multiply formally by $\Phi(y)$:

$$\frac{d}{dt}\int_0^\infty f(t,y)\Phi(y)dy =$$
$$\frac{1}{2}\int_0^\infty\int_0^\infty a(y,y')f(t,y)f(t,y')\tilde{\Phi}(y,y')dydy'$$
$$+\int_0^\infty f(t,y)y^\gamma\int_0^y B(\frac{y}{y'})\left(\Phi(y') - \frac{y'}{y}\Phi(y)\right)dydy'$$

where $\tilde{\Phi}(y,y') = \Phi(y+y') - \Phi(y) - \Phi(y')$.

Estimates of the moments. Taking $\Phi(y) = y^k$:

$$\frac{d}{dt}\int_0^\infty f(t,y)y^k dy =$$
$$\frac{1}{2}\int_0^\infty\int_0^\infty \Lambda_k(y,y')f(t,y)f(t,y')dydy'$$
$$+ C_{k,\gamma}(1-k)\int_0^\infty y^{\gamma+k+1}f(t,y)dy'$$

with

$$C_{k,\gamma} = \frac{1+\gamma}{1-k}\int_0^1 B(\sigma)(\sigma^k - \sigma)d\sigma > 0,$$

$$\Lambda_k(y,y') = (k-1)\left(y^\alpha\,(y')^\beta + y^\beta\,(y')^\alpha\right)\times$$
$$\times\left((y+y')^k - y^k - (y')^k\right) \geq 0.$$

We deduce that for $k > 1$

$$\frac{d}{dt}M_k \leq C_{k,1}\,M_{\beta-1+k}\,M_{1+\alpha} - C_{k,2}M_{1+\gamma+k},$$

and

$$\frac{d}{dt}M_\lambda \leq C_{\lambda,1}\,M_{1+\gamma+\lambda} - C_{\lambda,2}\,M_\lambda^2,$$

Using that $\alpha \leq 0$ and $\gamma \geq -1$ we finally obtain

$$\frac{d}{dt}(M_\lambda + M_{2-\beta}) + (M_\lambda + M_{2-\beta})^q \leq C_1 + C_2\,(M_\lambda + M_{2-\beta}).$$

Using Gronwall's Lemma we deduce:

$$\sup_{t\geq 0}(M_\lambda(t) + M_{2-\beta}(t)) \leq \max\left(C_0, M_\lambda(0) + M_{2-\beta}(0)\right)$$

We may then estimate M_k for $k > 2-\beta$ as

$$\frac{d}{dt}M_k \leq C_1\, M_k^{\theta_1} - C_2\, M_k^{\theta_2},$$

with $0 \leq \theta_1 \leq 1$ and $\theta_2 > \theta_1, \theta_2 \geq 1$. Using again the Gronwall's Lemma we conclude

$$\sup_{t\geq 0} M_k(t) \leq \max\left(C, M_k(0)\right), \text{ for all } k > 2-\beta.$$

A similar argument shows that

$$\sup_{t\geq 0} M_{-r}(t) \leq \max(C, M_{-r}(0)),\ \forall r \in [0,1).$$

$\underline{L^p \text{ estimates.}}$ Take $\Phi(y) = f^{p-1}(y)$. Then,

$$\int_0^\infty Q(f) f^{p-1} dy = \int_0^\infty \int_0^\infty y^\alpha y'^\beta f f' \times$$
$$\times \left((f'')^{p-1} - f^{p-1} - (f')^{p-1} \right) dy dy'$$
$$\leq -M_\beta \int_0^\infty y^\alpha f^p dy.$$

On the other hand,

$$\int_0^\infty L(f) f^{p-1} dy =$$
$$\int_0^\infty y^\gamma f \int_0^y B(\frac{y'}{y}) \left((f')^{p-1} - \frac{y'}{y} f^p \right) dy' dy$$
$$\leq ||B||_\infty M_\gamma \int_0^\infty f^{p-1} dy - \int_0^1 y B(y) dy \int_0^\infty y^{\gamma+1} f^p dy.$$

This gives

$$\frac{d}{dt} \int_0^\infty f^p dy + C_1 \int_0^\infty (y^\alpha + y^{\gamma+1}) f^p dy \leq$$
$$C_2 \left(\int_0^\infty f^p dy \right)^{\frac{p-2}{p-1}}$$

and using that $\alpha \leq 0$ or $\gamma + 1 = 0$ we deduce:

$$\frac{d}{dt}\int_0^\infty f^p dy + C_1 \int_0^\infty f^p dy \leq C_2 \left(\int_0^\infty f^p dy\right)^{\frac{p-2}{p-1}}$$

This gives:

$$\sup_{t\geq 0} ||f(t)||_p \leq \max(C, ||f_{in}||_p), \, \forall p \geq 2.$$

Fixed point argument.

$$Y = L^1_{2\alpha} \cap L^1_m, \quad m = \max(2\beta, 2-\beta, \gamma+1),$$

The semigroup of the coagulation fragmentation equation $S_t : Y \to Y$

$\mathcal{A}_k = \left\{f, M_1(f) = \rho, ||f||_2 \leq \mu_0, M^1_k(f(t)) \leq \mu_k\right\}$

By apriori estimates, $S_t : A_k \to A_k$ if μ_k large. Finally, define:

$$\mathcal{Z}_k = \bigcap_{\ell=1}^{k} A_k.$$

We obtain a sequence $(G_k)_{k>1}$ of steady states such that $G_k \in \mathcal{Z}_k$. A compactness argument gives a fixed point $g \in \bigcap_{k=1}^\infty \mathcal{Z}_k$.

Remark. If we only assume that $B \in \mathcal{M}_m$ for some $m \leq 2\alpha$, we only have that $g \in \bigcap_{k\geq 1} \mathcal{M}_k$.

5. Fragmentation: Asymptotic behaviour.

Take $f_{in} \in C_0(\mathbb{R}^+)$ such that $M_1(f_{in}) = \rho > 0$.
Let g_ρ: the unique profile of mass ρ.
Suppose that $f_{in} \neq g_\rho$.
Let f be the unique solution of:

$$\partial_t f = L(f), \quad f(0,x) = f_{in}.$$

The function

$$g(\tau, y) = e^{-\frac{2\tau}{1+\gamma}} f(e^\tau - 1, e^{-\frac{\tau}{1+\gamma}} y)$$

is the unique solution to

$$\frac{\partial g}{\partial \tau} = -2g - y\frac{\partial g}{\partial y} + (1+\gamma)L(g), \; g(0,y) = f_{in}.$$

(g_ρ is a stationary solution of this equation).
Then the function

$$H(g(\tau)) = \int_0^\infty y|g(\tau,y) - g_\rho(y)|dy$$

is non increasing in time:

$$\frac{d}{d\tau}H(g(\tau)) = D(h(\tau))$$

with $h(\tau, y) = g(\tau, y) - g_\rho(y)$ and

$$
\begin{aligned}
D(h) &= \int_0^\infty (-2h + yh_y + (1+\gamma)L(h))\, sign(h(y))dy \\
&= \int_0^\infty \int_0^y b(y, y')(h(y) sign(h(y') - |h(y)|)y'dy'dy \\
&\leq 0.
\end{aligned}
$$

If for $0 < \tau_1 < \tau_2$, $H(g(\tau_1)) = H(g(\tau_2))$then:

$$\int_{\tau_1}^{\tau_2} D(h(s))ds = 0$$

from where: $\quad D(h(s)) = 0, \quad \forall s \in (\tau_1, \tau_2)$.

$$h(s, y) sign(h(s, y') - |h(s, y)| = 0, \ \forall s \in (\tau_1, \tau_2)$$

$$sign h(s, y) = sign h(s, y'), \text{ for a.e. } y, y'$$

This implies that $sign h(s, y)$ is constant for $y \in \mathbb{R}^+$ and $s \in (\tau_1, \tau_2)$. Since

$$\int_0^\infty yh(s, y)dy = \int_0^\infty y(g(\tau, y) - g_\rho(y))dy = 0$$

this is impossible.

$H(g(\tau))$ is then a strict Lyapunov functional on L^1_1.

Since $f_{in} \in L^1_1 \cap BV_1$, $g(\tau) \in L^1_1 \cap BV_1$. The trajectory $(g(\tau))_{\tau>0}$ is then in a compact subset of L^1_1. This implies

$$\lim_{\tau \to +\infty} H(g(\tau)) = 0$$

or,

$$\lim_{\tau \to +\infty} \int_0^\infty y|e^{-\frac{2\tau}{1+\gamma}} f(e^\tau - 1, e^{-\frac{\tau}{1+\gamma}} y) - g_\rho(y)|dy = 0$$

$$\Leftrightarrow$$

$$\lim_{t \to +\infty} \int_0^\infty y|f(t-1,y) - t^{\frac{2}{1+\gamma}} g_\rho(t^{\frac{1}{1+\gamma}} y)|dy = 0.$$