Ariketa 1	Ariketa 2	Ariketa 3	Ariketa 4	Ariketa 5	1. zatia

Azterketaren iraupena: 3 ordu

IZEN-ABIZENAK:

TALDEA:

1.- Izan bedi $\begin{cases} a_1 = 1 \\ a_{n+1} = \dfrac{n}{n+1} \cdot a_n \quad \forall n \geq 1 \end{cases}$

a) Kalkulatu $\lim\limits_{n\to\infty} a_n$

b) Aztertu $\sum\limits_{n=1}^{\infty} a_n$ **seriearen izaera**

(2 puntu)

a)

$$\forall n \geq 1 \quad a_{n+1} = \frac{n}{n+1} \cdot a_n \quad \Rightarrow \begin{cases} a_2 = \dfrac{1}{2} \cdot a_1 = \dfrac{1}{2} \\ a_3 = \dfrac{2}{3} \cdot a_2 = \dfrac{2}{3} \cdot \dfrac{1}{2} = \dfrac{1}{3} \\ a_4 = \dfrac{3}{4} \cdot a_3 = \dfrac{3}{4} \cdot \dfrac{1}{3} = \dfrac{1}{4} \\ \quad \vdots \\ a_n = \dfrac{1}{n} \end{cases} \quad \Rightarrow \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

b) $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ dibergentea da.

2.- $\sum_{n=1}^{\infty} 2n \cdot x^{2n-1}$ **berretura-seriea emanik:**

a) Aurkitu bere konbergentzi arloa.

b) Kalkulatu bere batura $x = \frac{1}{2}$ **baliorako.**

(2 puntu)

a) Balio absolutuen serieari D´Alambert-en irizpidea aplikatuko diogu:

$$\lim_{n\to\infty} \frac{2(n+1)\cdot |x|^{2n+1}}{2n \cdot |x|^{2n-1}} = |x|^2 = x^2 < 1 \quad \Leftrightarrow \quad x \in (-1,1)$$

Baldin $x = 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} 2n$ dibergentea da

Baldin $x = -1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} -2n$ dibergentea da

Beraz, $\sum_{n=1}^{\infty} 2n \cdot x^{2n-1}$ konbergentea da (absolutuki) $\forall x \in (-1,1)$.

b) $\exists S(x) = \sum_{n=1}^{\infty} 2n \cdot x^{2n-1} \quad \forall x \in (-1,1)$. Eta integragarria da $[0, x]$ tartean $\forall x \in (-1,1)$:

$$\forall x \in (-1,1) \quad \int_0^x S(t)dt = \sum_{n=1}^{\infty} x^{2n} \overset{(1)}{=} \frac{x^2}{1-x^2} \quad \Rightarrow \quad S(x) = \left(\frac{x^2}{1-x^2} \right)' = \frac{2x}{(1-x^2)^2} \quad \forall x \in (-1,1)$$

$$\text{Baldin } x = \frac{1}{2} \quad \Rightarrow \quad S\left(\frac{1}{2}\right) = \frac{2 \cdot \frac{1}{2}}{\left(1 - \frac{1}{2^2}\right)^2} = \frac{1}{\left(\frac{3}{4}\right)^2} = \frac{16}{9}$$

(1) Serie geometrikoa da, arrazoia $r = x^2$ delarik.

3.- Aurkitu analitiko eta grafikoki hurrengo funtzioaren definizio-eremua:

$$f(x,y)=\sqrt{1-x^2\cdot y^2}+\mathrm{L}\left(|x|+|y|-1\right)+\mathrm{L}\left(3-|x|-|y|\right)$$

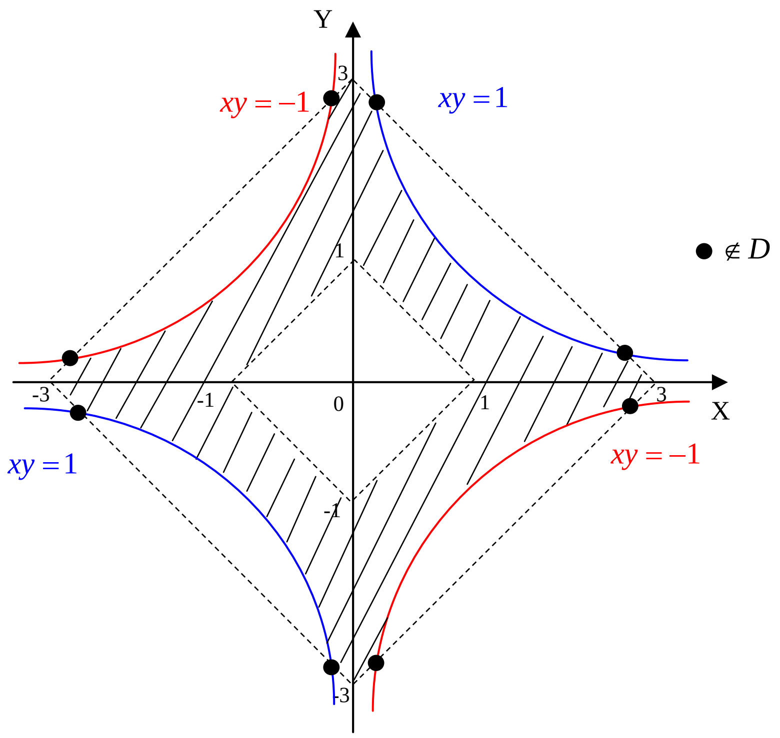
(2 puntu)

$$D=\left\{(x,y)\in\mathbb{R}^2 / 1-x^2\cdot y^2\geq 0,\ |x|+|y|-1>0,\ 3-|x|-|y|>0\right\}$$

$$1-x^2\cdot y^2\geq 0 \quad\Leftrightarrow\quad x^2\cdot y^2\leq 1 \quad\Leftrightarrow\quad |xy|\leq 1 \quad\Leftrightarrow\quad -1\leq xy\leq 1$$

$$|x|+|y|-1>0 \quad\Leftrightarrow\quad |x|+|y|>1$$

$$3-|x|-|y|>0 \quad\Leftrightarrow\quad |x|+|y|<3$$



4.- Izan bedi $f(x,y)=\begin{cases}\cos\left(\dfrac{xy^2}{x^2+y^2}\right) & \forall(x,y)\neq(0,0)\\ 1 & (x,y)=(0,0)\end{cases}$ **.**

a) Estudiatu *f*-ren jarraitutasuna (0,0) puntuan.

b) Kalkulatu $f'_x(0,0)$ eta $f'_y(0,0)$.

c) Aztertu *f* -ren diferentziagarritasuna (0,0) puntuan.

(2 puntu)

a) Jarraitutasuna (0,0) puntuan:

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\cos\left(\frac{xy^2}{x^2+y^2}\right)\overset{(\text{polarretan})}{=}\lim_{\rho\to0^+}\cos\left(\frac{\rho^3\cdot\cos\theta\cdot\sin^2\theta}{\rho^2}\right)=$$

$$=\lim_{\rho\to0^+}\cos\left(\rho\cdot\overbrace{\cos\theta\cdot\sin^2\theta}^{\text{mugatua}}\right)=\cos 0=1=f(0,0)$$

Beraz, *f* jarraitua da (0,0) puntuan.

b) $$f'_x(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{\cos\left(\frac{0}{h^2}\right)-1}{h}=\lim_{h\to0}\frac{1-1}{h}=0$$

$$f'_y(0,0)=\lim_{k\to0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to0}\frac{\cos\left(\frac{0}{k^2}\right)-1}{k}=\lim_{k\to0}\frac{1-1}{k}=0$$

c) Baldintza beharrezkoa eta nahikoa aplikatuko dugu.

Hots, *f* diferentziagarria da $(0,0)$ puntuan $\Leftrightarrow$

$$\Leftrightarrow\quad \lim_{(h,k)\to(0,0)}\frac{\left|f(h,k)-f(0,0)-h\cdot f'_x(0,0)-k\cdot f'_y(0,0)\right|}{\sqrt{h^2+k^2}}=0$$

Hau da:

$$\lim_{(h,k)\to(0,0)}\frac{\left|\cos\left(\frac{hk^2}{h^2+k^2}\right)-1\right|}{\sqrt{h^2+k^2}}\overset{(\text{polarretan})}{=}\lim_{\rho\to0^+}\frac{\left|\cos\left(\rho\cdot\cos\theta\cdot\sin^2\theta\right)-1\right|}{\rho}\sim\lim_{\rho\to0^+}\frac{\frac{\left(\rho\cdot\cos\theta\cdot\sin^2\theta\right)^2}{2}}{\rho}$$

$$=\lim_{\rho\to0^+}\frac{\rho^2\cdot\cos^2\theta\cdot\sin^4\theta}{2\rho}=\frac{1}{2}\cdot\lim_{\rho\to0^+}\rho\cdot\overbrace{\cos^2\theta\cdot\sin^4\theta}^{\text{mugatua}}=0\quad\forall\theta$$

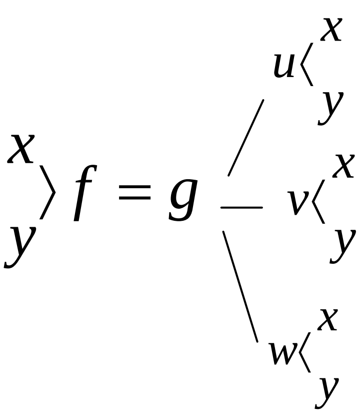
Beraz, *f* diferentziagarria da (0,0) puntuan.

5.- Izan bitez *f* eta *g* funtzio diferentziagarriak non $f(x,y)=g(u,v,w)$, eta $\begin{cases} u=2x+y \\ v=e^{2x+y} \\ w=\sin(2x+y) \end{cases}$ **. Baldin $\overrightarrow{\nabla g}(0,1,0)=(1,1,0)$:**

a) Kalkulatu $\overrightarrow{\nabla f}(0,0)$.

b) Kalkulatu $\left.\dfrac{df}{d\vec{h}}\right|_{(0,0)}$ non $\vec{h}=(-1,2)$. Justifikatu emaitza.

(2 puntu)



a) $\overrightarrow{\nabla f}(0,0)=\left(f'_x(0,0),\ f'_y(0,0)\right)$

$$f'_x=g'_u\cdot u'_x+g'_v\cdot v'_x+g'_w\cdot w'_x=2\cdot g'_u+2e^{2x+y}\cdot g'_v+2\cos(2x+y)\cdot g'_w$$
$$f'_y=g'_u\cdot u'_y+g'_v\cdot v'_y+g'_w\cdot w'_y=g'_u+e^{2x+y}\cdot g'_v+\cos(2x+y)\cdot g'_w$$

$$(x,y)=(0,0)\Rightarrow(u,v,w)=(0,1,0)$$

$$f'_x(0,0)=2\cdot g'_u(0,1,0)+2\cdot g'_v(0,1,0)+2\cdot g'_w(0,1,0)$$
$$f'_y(0,0)=g'_u(0,1,0)+g'_v(0,1,0)+g'_w(0,1,0)$$

$$\overrightarrow{\nabla g}(0,1,0)=(1,1,0)\quad\Rightarrow\quad g'_u(0,1,0)=1\quad g'_v(0,1,0)=1\quad g'_w(0,1,0)=0\quad\Rightarrow$$

$$\left.\begin{array}{l} f'_x(0,0)=2+2+0=4 \\ f'_y(0,0)=1+1+0=2 \end{array}\right\}\quad\Rightarrow\quad\overrightarrow{\nabla f}(0,0)=(4,2)$$

b) $\vec{h}=(-1,2)\quad\Rightarrow\quad\left|\vec{h}\right|=\sqrt{5}\quad\Rightarrow\quad\vec{h}=\left(\dfrac{-1}{\sqrt{5}},\dfrac{2}{\sqrt{5}}\right)$ unitarioa.

$$\left.\frac{df}{d\vec{h}}\right|_{(0,0)}=f'_x(0,0)\cdot\left(\frac{-1}{\sqrt{5}}\right)+f'_y(0,0)\cdot\frac{2}{\sqrt{5}}=4\cdot\left(\frac{-1}{\sqrt{5}}\right)+2\cdot\frac{2}{\sqrt{5}}=0$$

$\vec{h}=(-1,2)$ -ren norabidean *f*-ren aldakuntza nulua da $\vec{h}=(-1,2)\perp\overrightarrow{\nabla f}(0,0)=(4,2)$ baitira.

Ariketa 6	Ariketa 7	Ariketa 8	Ariketa 9	2. zatia

Azterketaren iraupena: 3 ordu

IZEN-ABIZENAK:

TALDEA:

6.- Aurkitu $f(x,y) = xy + 2x - \mathrm{L}\left(x^2 \cdot y\right)$ **funtzioaren puntu kritikoa planoko eskualdean non** $x > 0$ **eta** $y > 0$**, eta estudiatu zein motatako muturra den.**

(2 puntu)

Puntu kritikoa:

$$\left.\begin{array}{l} f'_x = y + 2 - \dfrac{2xy}{x^2 \cdot y} = y + 2 - \dfrac{2}{x} = 0 \\ f'_y = x - \dfrac{x^2}{x^2 \cdot y} = x - \dfrac{1}{y} = 0 \quad \Leftrightarrow \quad x = \dfrac{1}{y} \end{array}\right\} \Rightarrow \quad y + 2 - 2y = 0 \quad \Leftrightarrow \quad y = 2 \quad \Leftrightarrow \quad x = \frac{1}{2}$$

Beraz, $P\left(\frac{1}{2}, 2\right)$ puntu kritikoa da.

Zein motatako muturra den lortzeko bigarren diferentzialaren zeinua aztertuko dugu:

$$\left.\begin{array}{l} f''_{x^2} = \dfrac{2}{x^2} \quad \Rightarrow \quad f''_{x^2}(P) = 8 \\ f''_{xy} = 1 \quad \Rightarrow \quad f''_{x^2}(P) = 1 \\ f''_{y^2} = \dfrac{1}{y^2} \quad \Rightarrow \quad f''_{x^2}(P) = \dfrac{1}{4} \end{array}\right\} \Rightarrow \quad d^2 f(P) = 8(dx)^2 + 2dxdy + \frac{1}{4}(dy)^2 =$$

$$= \left(2\sqrt{2}dx + \frac{1}{2\sqrt{2}}dy\right)^2 + \frac{1}{8}(dy)^2 > 0 \quad \Rightarrow$$

P minimo erlatiboa da.

7.- Aztertu $I=\int_1^{\infty}\frac{dx}{x^2\left[\arctan(x-1)\right]^2}$ **integral inpropioaren izaera.**

(2 puntu)

$I=\int_1^{\infty} f(x)dx$ non $f(x)=\frac{1}{x^2\left[\arctan(x-1)\right]^2}>0 \quad \forall x\in(1,\infty)$

∞ puntu singularra da.

$\lim\limits_{x\to 1^+} f(x)\sim \lim\limits_{x\to 1^+}\frac{1}{x^2\cdot(x-1)^2}=\infty \quad\Rightarrow\quad x=1$ ere puntu singularra da. Orduan:

$$I=\int_1^{\infty} f(x)dx=\int_1^{a} f(x)dx+\int_a^{\infty} f(x)dx=I_1+I_2\ ,\quad 1<a<\infty$$

I konbergentea da $\quad\Leftrightarrow\quad I_1$ eta I_2 konbergenteak dira

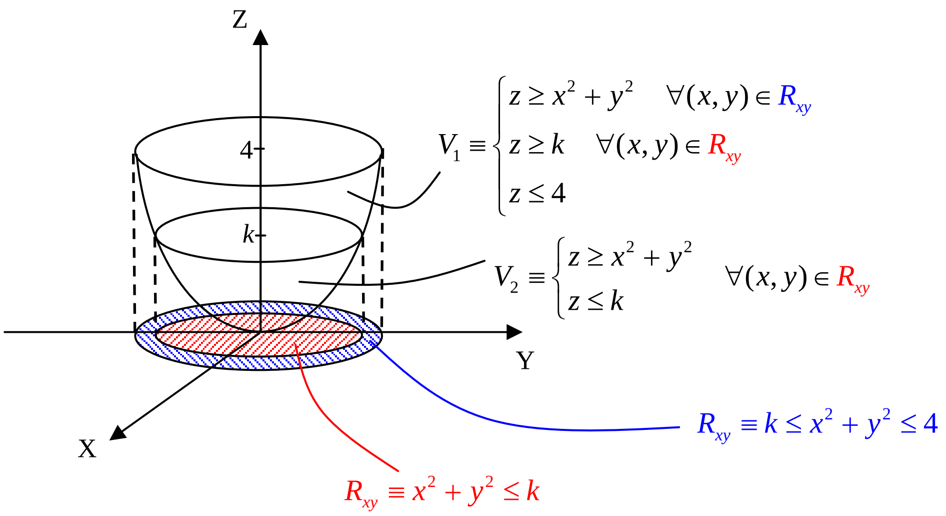
Konparaziozko irizpidea erabiliko dugu.

$\boxed{I_1}$ aztertzeko integral eredua $\int_1^{a}\frac{dx}{(x-1)^m}$, $m>0$, $\begin{cases}\text{konbergentea } \forall m<1\\ \text{dibergentea } \forall m\geq 1\end{cases}$ **:**

$\lim\limits_{x\to 1^+}\frac{f(x)}{\frac{1}{(x-1)^m}}=\lim\limits_{x\to 1^+}\frac{(x-1)^m}{x^2\cdot\left[\arctan(x-1)\right]^2}\sim\lim\limits_{x\to 1^+}\frac{(x-1)^m}{(x-1)^2}\underset{(m=2>1)}{=}1\in(0,\infty)\quad\Rightarrow\quad I_1$ dibergentea da $\quad\Rightarrow\quad I$ dibergentea da.

8.- $V \equiv \begin{cases} z \geq x^2 + y^2 \\ z \leq 4 \end{cases}$ **solidoa emanik, aurkitu** $k > 0$ **konstantearen balioa zeinerako** $z = k$ **planoak solido hori bolumen bereko bi zatitan banatzen duen.**

(2 puntu)



$V = V_1 + V_2 \quad R_{xy} \equiv R_{xy} + R_{xy} \equiv x^2 + y^2 \leq 4$

$z = k$ planoa kalkulatu behar dugu, non $V_1 = V_2$ edo $V = 2 \cdot V_2$

$$\text{Bolumena}(V) = \iint_{R_{xy}} \left[4 - \left(x^2 + y^2 \right) \right] dxdy \overset{(*)}{=} \int_0^{2\pi} \int_0^2 \rho \left(4 - \rho^2 \right) d\rho d\theta = 2\pi \left[2\rho^2 - \frac{\rho^4}{4} \right]_0^2 = 8\pi$$

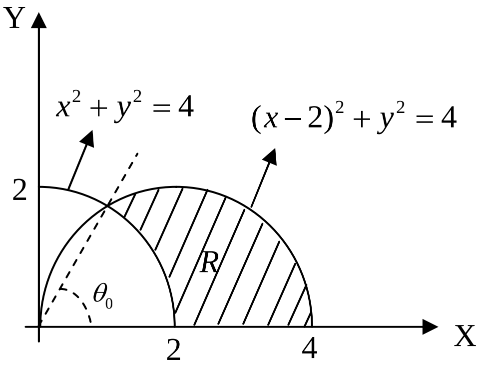
$$\text{Bolumena}(V_2) = \iint_{R_{xy}} \left[k - \left(x^2 + y^2 \right) \right] dxdy \overset{(*)}{=} \int_0^{2\pi} \int_0^{\sqrt{k}} \rho \left(k - \rho^2 \right) d\rho d\theta = 2\pi \left[\frac{k\rho^2}{2} - \frac{\rho^4}{4} \right]_0^{\sqrt{k}} = \frac{\pi k^2}{2}$$

Orduan, $V = 2 \cdot V_2 \quad \Leftrightarrow \quad 8\pi = \pi k^2 \quad \Leftrightarrow \quad k = 2\sqrt{2}$

(*) Polarretan $\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \quad |J| = \rho \quad \Rightarrow \quad R_{xy} \equiv \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq 2 \end{cases} \quad R_{xy} \equiv \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq \sqrt{k} \end{cases}$

9.- Kalkulatu $\iint_R \frac{y}{x^2+y^2}\,dxdy$, non *R* planoko lehenengo koadranteko eskualdea den, $x^2-4x+y^2=0$ kurbaren barrutik eta $x^2+y^2=4$ kurbaren kanpotik definiturikoa.

(2 puntu)



$$x^2-4x+y^2=0 \quad \Leftrightarrow \quad (x-2)^2+y^2=4$$

$$\left.\begin{array}{l} x^2-4x+y^2=0 \\ x^2+y^2=4 \end{array}\right\} \Rightarrow \quad 4x=4 \quad \Leftrightarrow \quad x=1 \overset{\text{1. koadrantean}}{\Rightarrow} y=\sqrt{3} \quad \Rightarrow \quad \theta_0=\frac{\pi}{3}$$

Polarretan

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \quad |J|=\rho \quad \Rightarrow \quad \begin{cases} x^2+y^2=4 \Leftrightarrow \rho=2 \\ x^2+y^2=4x \Leftrightarrow \rho=4\cos\theta \end{cases} \quad \Rightarrow \quad R\equiv\begin{cases} 0\le\theta\le\frac{\pi}{3} \\ 2\le\rho\le 4\cos\theta \end{cases}$$

Orduan:

$$\iint_R \frac{y}{x^2+y^2}\,dxdy = \int_0^{\frac{\pi}{3}}\int_2^{4\cos\theta} \rho\frac{\rho\sin\theta}{\rho^2}\,d\rho d\theta = \int_0^{\frac{\pi}{3}}\int_2^{4\cos\theta} \sin\theta d\rho d\theta = \int_0^{\frac{\pi}{3}}(4\sin\theta\cos\theta-2\sin\theta)d\theta =$$

$$=\left[2\sin^2\theta+2\cos\theta\right]_0^{\frac{\pi}{3}} = 2\left(\frac{3}{4}+\frac{1}{2}-1\right)=\frac{1}{2}$$