Ariketa 1	Ariketa 2	Ariketa 3	Ariketa 4	Ariketa 5	Ariketa 6	Ariketa 7	Guztira

Azterketaren iraupena (1. zatia): 2 ordu

IZEN-ABIZENAK:

TALDEA:

1. ZATIA

1.- $\lim_{n\to\infty} a_n = 0$ **ezagutuz, kalkulatu** $\lim_{n\to\infty}\left(\dfrac{2-a_n}{2-a_n-(a_n)^2}\right)^{\frac{1}{\mathrm{L}\left(1+\tan^2(a_n)\right)}}$ **.**

(Puntu 1)

$$\lim_{n\to\infty}\left(\frac{2-a_n}{2-a_n-(a_n)^2}\right)^{\frac{1}{\mathrm{L}\left(1+\tan^2(a_n)\right)}} = 1^\infty = A \quad\Leftrightarrow\quad \mathrm{L}A = \lim_{n\to\infty}\frac{1}{\mathrm{L}\left(1+\tan^2(a_n)\right)}\cdot\mathrm{L}\left(\frac{2-a_n}{2-a_n-(a_n)^2}\right) =$$

$$= \lim_{n\to\infty}\frac{1}{\tan^2(a_n)}\cdot\left(\frac{2-a_n}{2-a_n-(a_n)^2}-1\right) = \lim_{n\to\infty}\frac{1}{(a_n)^2}\cdot\frac{2-a_n-2+a_n+(a_n)^2}{2-a_n-(a_n)^2} =$$

$$= \lim_{n\to\infty}\frac{1}{(a_n)^2}\cdot\frac{(a_n)^2}{2-a_n-(a_n)^2} = \lim_{n\to\infty}\frac{1}{2-a_n-(a_n)^2} = \frac{1}{2} \quad\Leftrightarrow\quad A = e^{1/2}$$

2.- Aztertu $\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ seriearen izaera.

(Puntu 1)

$\sum_{n=1}^{\infty} a_n$ non $a_n = \sqrt{n+1}-\sqrt{n} > 0 \quad \forall n$

Konbergentzi baldintza beharrezkoa erabiliz:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty}\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{2\sqrt{n}} = 0$$

Kasu honetan, BB betetzen da. Baina, aldi berean, $a_n = \sqrt{n+1}-\sqrt{n} \sim \frac{1}{2\sqrt{n}}$ ere betetzen dela frogatu dugu. Eta $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$, Riemann-en seriea da, kasu honetan dibergentea. Beraz, konparaziozko irizpidearen ondorioz, $\sum_{n=1}^{\infty} a_n$ dibergentea da.

3.- a) Aztertu $\sum_{n=0}^{\infty}\frac{e^{(n+2)b}}{3^{n+2}\cdot(n+2)}$ seriearen izaera, $b\in\mathbb{R}$ parametroaren balioen arabera.

b) Kalkulatu $\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}$ berretura-seriearen batura, konbergentzi arloa non balio duen adieraziz.

c) Aurreko bi ataletan lortutako emaitzak erabiliz, aurkitu a) ataleko seriearen batura $b=1$ eta $b=2$ balioetarako.

(3 puntu)

a) $\sum_{n=1}^{\infty}a_n$ non $a_n=\frac{e^{(n+2)b}}{3^{n+2}\cdot(n+2)}>0\quad\forall n$ eta $\forall b\in\mathbb{R}$

D´Alembert-en irizpidea erabiliz:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{e^{(n+3)b}}{3^{n+3}\cdot(n+3)}\cdot\frac{3^{n+2}\cdot(n+2)}{e^{(n+2)b}}=\frac{e^b}{3}\begin{cases}<1 & \Leftrightarrow\quad e^b<3\quad\Leftrightarrow\quad b<\mathrm{L}3\\ >1 & \Leftrightarrow\quad e^b>3\quad\Leftrightarrow\quad b>\mathrm{L}3\\ =1 & \Leftrightarrow\quad e^b=3\quad\Leftrightarrow\quad b=\mathrm{L}3\end{cases}$$

Beraz, $\forall b<\mathrm{L}3,\ \sum_{n=1}^{\infty}a_n$ konbergentea da, eta, $\forall b>\mathrm{L}3,\ \sum_{n=1}^{\infty}a_n$ dibergentea da.

$$b=\mathrm{L}3\quad\Leftrightarrow\quad e^b=3\quad\Rightarrow\quad a_n=\frac{e^{(n+2)b}}{3^{n+2}\cdot(n+2)}=\frac{3^{n+2}}{3^{n+2}\cdot(n+2)}=\frac{1}{n+2}\sim\frac{1}{n}$$

Eta $\sum_{n=1}^{\infty}\frac{1}{n}$ dibergentea da, orduan $\sum_{n=1}^{\infty}a_n$ dibergentea da.

Hau da, $\sum_{n=1}^{\infty}a_n\begin{cases}\text{konbergentea da }\forall b<\mathrm{L}3\\ \text{dibergentea da }\forall b\geq\mathrm{L}3\end{cases}$

b) Dakigunez, $\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}$ berretura-seriea konbergentea da jatorriarekiko tarte simetrikoan, beraz, $\exists S(x)=\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}\quad\forall x\in(-R,R)$, non $0\leq R\leq\infty$ izan daiteke.

Eta, badakigu ere, deribagarria dela $\forall x\in(-R,R)$. Hau da:

$$\exists S'(x)=\sum_{n=0}^{\infty}x^{n+1}\overset{(1)}{=}\frac{x}{1-x}\quad\forall x\in(-1,1)\ \text{(hau da, } R=1\text{)}$$

(1) Serie geometrikoa da, $r=x$ delarik beraz, konbergentea da $\Leftrightarrow\quad |r|=|x|<1$

Eta, emaitza hori integratuz $\forall x\in(-1,1)$:

$$S(x)=\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}=\int\frac{x}{1-x}dx=\int\left(-1+\frac{1}{1-x}\right)dx=-x-\mathrm{L}(1-x)+k\overset{(2)}{=}-x-\mathrm{L}(1-x)$$

(2) $x=0$ puntuan ordezkatuz: $S(0)=0 \quad \Rightarrow \quad k=0$

Beraz, $S(x)=\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}=-x-\mathrm{L}(1-x) \quad \forall x\in(-1,1)$

Eta orain, tarte horretako mugak aztertu behar ditugu:

$x=1$ puntuan, $\sum_{n=0}^{\infty}\frac{1}{n+2}$ serie dibergentea dugu $\Rightarrow \nexists S$

$x=-1$ puntuan, $\begin{cases}\sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{n+2} \text{ bald. konbergentea da (Leibniz)} \Rightarrow \exists S\text{, eta jarraitua da} \\ \exists f(x)=-x-\mathrm{L}(1-x)\text{, eta jarraitua da}\end{cases}$

eta $S(x)=f(x) \quad \forall x\in(-1,1)$, orduan:

$$S(x)=\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}=-x-\mathrm{L}(1-x) \quad \forall x\in\left[-1,1\right)$$

c) $b=1<\mathrm{L}3 \quad \Rightarrow \quad \sum_{n=1}^{\infty}a_n=\sum_{n=0}^{\infty}\frac{e^{n+2}}{3^{n+2}\cdot(n+2)}=\sum_{n=0}^{\infty}\frac{(e/3)^{n+2}}{n+2}$ konbergentea da, beraz, batura finitua dauka. Eta, ikusten denez, b) ataleko berretura-seriean $x=\frac{e}{3}$ egiten badugu, serie hori lortzen da. Beraz, $\sum_{n=0}^{\infty}\frac{(e/3)^{n+2}}{n+2}=S\left(\frac{e}{3}\right)=-\frac{e}{3}-\mathrm{L}\left(1-\frac{e}{3}\right)$.

$b=2>\mathrm{L}3 \quad \Rightarrow \quad \sum_{n=1}^{\infty}a_n$ dibergentea da $\Rightarrow \sum_{n=1}^{\infty}a_n=\infty$

4.- Aurkitu analitiko eta grafikoki $f(x,y)=\dfrac{\sqrt{\left(1-e^{x+y}\right)\left(e^{x-y}-1\right)}}{\mathrm{L}\left(4-x^2-y^2\right)}$ **funtzioaren definizio-eremua.**

(1.5 puntu)

$$D=\left\{(x,y)\in\mathbb{R}^2 \,/\, \left(1-e^{x+y}\right)\left(e^{x-y}-1\right)\geq 0,\ \mathrm{L}\left(4-x^2-y^2\right)\neq 0,\ 4-x^2-y^2>0\right\}$$

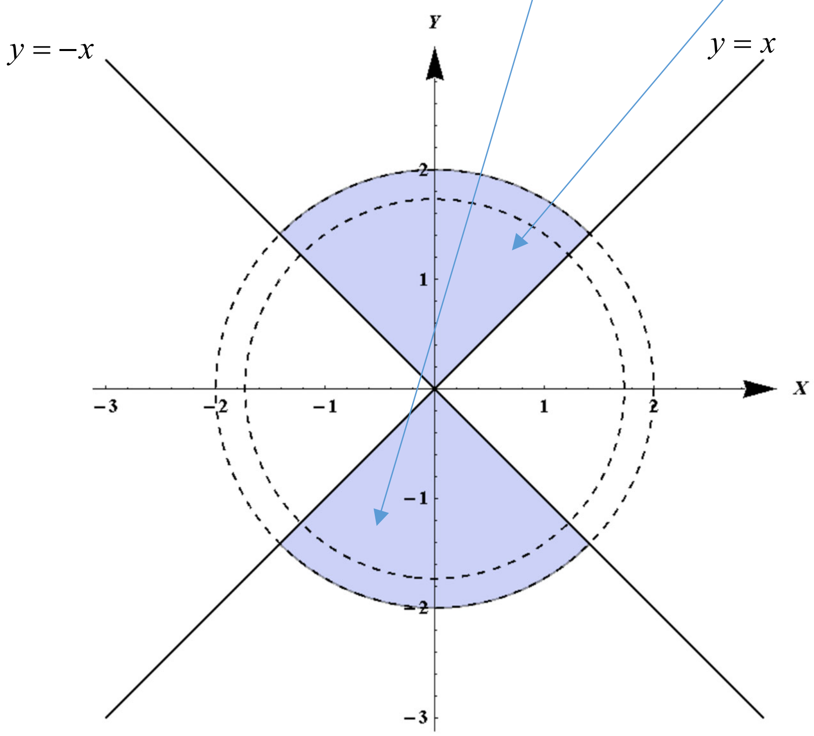
- $\left(1-e^{x+y}\right)\left(e^{x-y}-1\right)\geq 0 \ \Rightarrow\ \left(1-e^{x+y}\right)\left(e^{x-y}-1\right)=0 \ \text{ edo } \ \left(1-e^{x+y}\right)\left(e^{x-y}-1\right)>0$

$$\left(1-e^{x+y}\right)\left(e^{x-y}-1\right)=0 \ \Rightarrow\ \begin{cases}1-e^{x+y}=0\\ \text{edo}\\ e^{x-y}-1=0\end{cases} \Leftrightarrow \begin{cases}e^{x+y}=1\\ \text{edo}\\ e^{x-y}=1\end{cases} \Leftrightarrow \begin{cases}x+y=0\\ \text{edo}\\ x-y=0\end{cases}$$

$$\left(1-e^{x+y}\right)\left(e^{x-y}-1\right)>0 \ \Rightarrow\ \begin{cases}1-e^{x+y}>0\\ e^{x-y}-1>0\end{cases} \text{ edo } \begin{cases}1-e^{x+y}<0\\ e^{x-y}-1<0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases}e^{x+y}<1\\ e^{x-y}>1\end{cases} \text{ edo } \begin{cases}e^{x+y}>1\\ e^{x-y}<1\end{cases} \Leftrightarrow \begin{cases}x+y<0\\ x-y>0\end{cases} \text{edo} \begin{cases}x+y>0\\ x-y<0\end{cases}$$

- $\mathrm{L}\left(4-x^2-y^2\right)\neq 0 \ \Leftrightarrow\ 4-x^2-y^2\neq 1 \ \Leftrightarrow\ x^2+y^2\neq 3$
- $4-x^2-y^2>0 \ \Leftrightarrow\ x^2+y^2<4$



5.- $f(x,y)=\begin{cases} \dfrac{e^{x^2y}-1}{x^2+y^2} & \forall (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$ **funtzioa emanik,**

a) **Aztertu bere jarraitutasuna (0,0) puntuan.**
b) **Kalkulatu bere deribatu partzialak (0,0) puntuan.**
c) **Aztertu bere diferentziagarritasuna (0,0) puntuan.**
d) **Jarraituak dira *f*-ren deribatu partzialak (0,0) puntuan? Erantzuna arrazoitu.**

(2 puntu)

a) $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{e^{x^2y}-1}{x^2+y^2} \overset{(1)}{=} \lim_{\substack{\rho\to0^+\\ \forall\theta}} \frac{e^{\rho^3\cos^2\theta\cdot\sin\theta}-1}{\rho^2} = \lim_{\substack{\rho\to0^+\\ \forall\theta}} \frac{\rho^3\cos^2\theta\cdot\sin\theta}{\rho^2} =$$

$$= \lim_{\substack{\rho\to0^+\\ \forall\theta}} \rho\cos^2\theta\cdot\sin\theta = 0 = f(0,0) \quad \Leftrightarrow \quad f \text{ jarraitua da (0,0) puntuan.}$$

(1) polarretan adieraziz: $\begin{cases} x=\rho\cdot\cos\theta \\ y=\rho\cdot\sin\theta \end{cases} \quad \forall\theta\in[0,2\pi)$

b) $$f'_x(0,0) = \lim_{h\to0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to0}\frac{\frac{e^0-1}{h^2}-0}{h} = \lim_{h\to0}\frac{0}{h^3}=0$$

$$f'_y(0,0) = \lim_{k\to0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to0}\frac{\frac{e^0-1}{k^2}-0}{k} = \lim_{k\to0}\frac{0}{k^3}=0$$

c) BBN aplikatuko dugu:

f diferentziagarria (0,0) puntuan $\Leftrightarrow \displaystyle\lim_{(h,k)\to(0,0)} \frac{\left|f(h,k)-f(0,0)-h\cdot f'_x(0,0)-k\cdot f'_y(0,0)\right|}{\sqrt{h^2+k^2}}=0$

$$\lim_{(h,k)\to(0,0)} \frac{\left|f(h,k)-f(0,0)-h\cdot f'_x(0,0)-k\cdot f'_y(0,0)\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\left|\frac{e^{h^2k}-1}{h^2+k^2}\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\left|e^{h^2k}-1\right|}{\left(h^2+k^2\right)^{3/2}} \overset{(2)}{=}$$

(2) polarretan adieraziz: $\begin{cases} h=\rho\cdot\cos\theta \\ k=\rho\cdot\sin\theta \end{cases} \quad \forall\theta\in[0,2\pi)$

$$\lim_{\substack{\rho\to0^+\\ \forall\theta}} \frac{\left|e^{\rho^3\cos^2\theta\cdot\sin\theta}-1\right|}{\rho^3} = \lim_{\substack{\rho\to0^+\\ \forall\theta}} \frac{\rho^3\cos^2\theta\cdot|\sin\theta|}{\rho^3} = \cos^2\theta\cdot|\sin\theta|\neq0 \quad \Rightarrow \quad f \text{ ez da}$$

diferentziagarria (0,0) puntuan.

d) *f*-ren deribatu partzialak jarraituak balira (0,0) puntuan, *f* diferentziagarria litzateke puntu horretan. Beraz, ezin dira jarraituak izan.

6.- Aurkitu $f(x) = x \cdot e^{\sqrt{x^2+x^4}}$ funtzioaren deribatua $\forall x \in \mathbb{R}$.

(0.5 puntu)

$\forall x \neq 0$, f funtzio deribagarrien konposaketa da, beraz deribazio-erregelak erabil daitezke bere deribatua lortzeko:

$$f'(x) = e^{\sqrt{x^2+x^4}} + x \cdot \frac{2x+4x^3}{2\sqrt{x^2+x^4}} \cdot e^{\sqrt{x^2+x^4}} = e^{\sqrt{x^2+x^4}} \left(1 + \frac{x^2(1+2x^2)}{\sqrt{x^2+x^4}} \right) \quad \forall x \neq 0$$

$x = 0$ puntuan, berriz, $\sqrt{x}$ funtzio elementala ez da deribagarria, beraz, puntu horretako deribatua definizioz kalkulatu behar da:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \cdot e^{\sqrt{h^2+h^4}} - 0}{h} = \lim_{h \to 0} e^{\sqrt{h^2+h^4}} = 1$$

Beraz, $f'(x) = \begin{cases} e^{\sqrt{x^2+x^4}} \left(1 + \dfrac{x^2(1+2x^2)}{\sqrt{x^2+x^4}} \right) & \forall x \neq 0 \\ 1 & x = 0 \end{cases}$

7.- $\int_{-1}^{\infty} \frac{1}{x^2}\,dx$ **integrala emanik, adierazi zeintzuk diren bere puntu singularrak, eta kalkulatu.**

(Puntu 1)

$\int_{-1}^{\infty} \frac{1}{x^2}\,dx = \int_{-1}^{\infty} f(x)\,dx$ integral inpropioak bi puntu singular ditu:

∞ (integrazio-tartea infinitua da).

$x = 0$, *f* mugatua ez baitago puntu horretan $\left(\lim_{x\to 0} f(x) = \infty\right)$.

Kalkulatzeko, orduan, hiru zatitan deskonposatu beharko dugu:

$$I = \int_{-1}^{\infty} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx = I_1 + I_2 + I_3$$

Eta *I* konbergentea da $\Leftrightarrow I_1, I_2$ eta I_3 konbergenteak dira.

$I_1 = \int_{-1}^{0} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{0} = \infty \quad \Leftrightarrow \quad I_1$ dibergentea da.

Beraz, *I* dibergentea da: $\int_{-1}^{\infty} \frac{1}{x^2}\,dx = \infty$

Ariketa 1	Ariketa 2	Ariketa 3	Ariketa 4	Ariketa 5	Ariketa 6	Guztira

Azterketaren iraupena (2. zatia): 2 ordu eta erdi

IZEN-ABIZENAK:

TALDEA:

2. ZATIA

1.- $z(x,y)=x^2\cdot f(u)+g(v,w)$ **funtzio diferentziagarria emanik (*f* eta *g* ere funtzio diferentziagarriak izanik), non** $\begin{cases} u=3x+y \\ v=2xy \\ w=y^2 \end{cases}$ **, eta,** $\begin{cases} f(4)=3 \\ f'(4)=1 \\ g(2,1)=-1 \\ g'_v(2,1)=2 \\ g'_w(2,1)=-2 \end{cases}$ **ezagutuz, kalkulatu** $\overrightarrow{\nabla} z(1,1)$ **.**

(Puntu 1)

$$\overrightarrow{\nabla} z(1,1)=\left(z'_x(1,1),\, z'_y(1,1)\right)$$

$$z'_x=2x\cdot f(u)+x^2\cdot f'(u)\cdot u'_x+g'_v(v,w)\cdot v'_x=2x\cdot f(u)+3x^2\cdot f'(u)+2y\cdot g'_v(v,w)$$

$$z'_y=x^2\cdot f'(u)\cdot u'_y+g'_v(v,w)\cdot v'_y+g'_w(v,w)\cdot w'=x^2\cdot f'(u)+2x\cdot g'_v(v,w)+2y\cdot g'_w(v,w)$$

Eta, (1,1) puntuan ordezkatuz:

$$z'_x(1,1)=2\cdot f(4)+3\cdot f'(4)+2\cdot g'_v(2,1)=6+3+4=13$$

$$z'_y(1,1)=f'(4)+2\cdot g'_v(2,1)+2\cdot g'_w(2,1)=1+4-4=1$$

Beraz, $\overrightarrow{\nabla} z(1,1)=(13,1)$

2.- $F(x,y,z)=xy+z+\sin(2z)+\int_{1/x}^{1}\mathrm{L}(tx)dt+\int_{0}^{1}\frac{t^y-1}{\mathrm{L}(t)}dt$ **funtzioa eta** $P(x,y,z)=(1,0,0)$ **puntua emanik,**

a) Frogatu $F(x,y,z)=0$ **ekuazioak** $z=z(x,y)$ **funtzio inplizitua definitzen duela *P* puntuaren ingurunean.**

b) Kalkulatu z'_x **eta** z'_y**, (1,0) puntuan**

(1.75 puntu)

a) $F(x,y,z)=0$ ekuazioari, $P(x,y,z)=(1,0,0)$ puntuan, funtzio inplizituaren teorema aplikatuko diogu:

i. $F(P)=0$

ii. F-ren deribatu partzialak existitzen eta jarraituak dira $P(x,y,z)=(0,0,0)$ puntuaren ingurunean (non $x\neq 0$ eta $y\neq -1$):

$$\begin{cases} F'_x = y+\displaystyle\int_{1/x}^{1}\frac{t}{tx}dt+\frac{1}{x^2}\mathrm{L}\left(\frac{x}{x}\right)=y+\int_{1/x}^{1}\frac{dt}{x}=y+\left.\frac{t}{x}\right|_{1/x}^{1}=y+\frac{1}{x}-\frac{1}{x^2} \\ F'_y = x+\displaystyle\int_{0}^{1}\frac{\mathrm{L}(t)\cdot t^y}{\mathrm{L}(t)}dt=x+\int_0^1 t^y dt=x+\left.\frac{t^{y+1}}{y+1}\right|_0^1=x+\frac{1}{y+1} \\ F'_z=1+2\cos(2z) \end{cases}.$$

iii. $F'_z(P)=1+2=3\neq 0$

Beraz, $P(x,y,z)=(1,0,0)$ puntuaren ingurunean $\exists! z=z(x,y)$ diferentziagarria, $z(1,0)=0$ izanik.

b) $F(x,y,z(x,y))=0$ ekuazioa x-rekiko eta y-rekiko deribatuz:

$$y+\frac{1}{x}-\frac{1}{x^2}+\big(1+2\cos(2z)\big)\cdot z'_x=0 \quad \overset{\text{P puntuan}}{\Rightarrow} \quad z'_x(1,0)=0$$

$$x+\frac{1}{y+1}+\big(1+2\cos(2z)\big)\cdot z'_y=0 \quad \overset{\text{P puntuan}}{\Rightarrow} \quad 1+1+3z'_y(1,0)=0 \quad \Leftrightarrow \quad z'_y(1,0)=-\frac{2}{3}$$

3.- Kalkulatu $\begin{cases} z = x^2 + y^2 \\ 2y + z = 3 \end{cases}$ elipsetik (0,0,0) puntura dauden distantzia maximoa eta minimoa.

(1.5 puntu)

Distantzia hori $d(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ funtzioak ematen du, non (x, y, z) elipse horretako puntua den. Beraz, d funtzioaren maximo eta minimo erlatibo baldintzatuak kalkulatu behar ditugu, baldintzak elipsea definitzen duten ekuazioak direlarik.

Bestalde, elipsea multzo itxia eta mugatua da, eta d funtzio jarraitua, hortaz, Weierstrass-en teorema egiaztatzen da, eta, multzo horretan d funtzioak maximo eta minimo absolutuak dituela ziurtatuta daukagu.

Badakigu, gainera, $d(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ eta $f(x,y,z) = x^2 + y^2 + z^2$ funtzioek mutur berdinak dituztela, orduan, elipsean f funtzioaren mutur absolutuak kalkulatuko ditugu, eta, horiek emango dizkigute, hain zuzen ere, lortu nahi ditugun distantzia maximo eta minimoa.

Lagrange-ren biderkatzaileen metodoa erabiliz, hurrengo funtzioa definituko dugu:

$$w(x,y,z) = x^2 + y^2 + z^2 + \lambda\left(x^2 + y^2 - z\right) + \mu\left(2y + z - 3\right)$$

Eta, bere puntu kritikoak kalkulatuko ditugu:

$$\begin{cases} w'_x = 2x + 2\lambda x = 0 \iff 2x(1+\lambda) = 0 \Rightarrow \begin{cases} \lambda = -1 \Rightarrow \mu = 0 \Rightarrow z = -\dfrac{1}{2} = x^2 + y^2 \ \# \\ x = 0 \Rightarrow y^2 = z = 3 - 2y \iff y^2 + 2y - 3 = 0 \\ \qquad\qquad \Downarrow \\ \qquad \begin{cases} y = 1 \Rightarrow z = 1 \\ y = -3 \Rightarrow z = 9 \end{cases} \end{cases} \\ w'_y = 2y + 2\lambda y + 2\mu = 0 \\ w'_z = 2z - \lambda + \mu = 0 \\ x^2 + y^2 = z \\ 2y + z = 3 \iff z = 3 - 2y \end{cases}$$

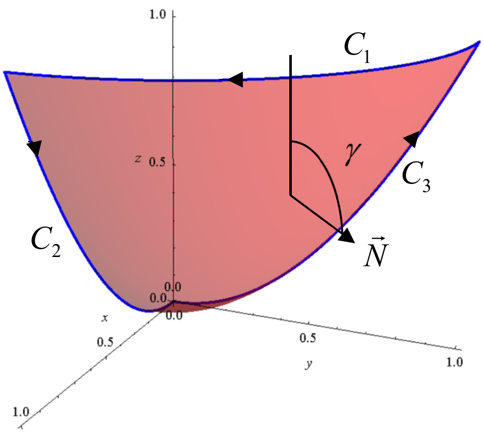
Beraz, bi puntu kritiko ditugu, $A = (0,1,1)$ eta $B = (0,-3,9)$. Orain, bi puntu hauetarako d distantzia kalkulatuz, $d(A) = \sqrt{2}$ distantzia minimoa eta $d(B) = \sqrt{90}$ distantzia maximoa dira.

4.- Izan bedi *S* gainazala, lehenengo oktantean $z=1$ planoak mugaturiko $z=x^2+y^2$ paraboloidearen zatia. Eta izan bedi *C* kurba itxia *S*-ren muga. $\vec{F}(x,y,z)=y\,\vec{i}+z\,\vec{j}+x\,\vec{k}$ eremu bektoriala emanik:

a) Kalkulatu $\iint_S \vec{F}\cdot d\vec{S}$, *S* gainazalaren kanpoko aurpegitik.

b) Kalkulatu $\oint_C \vec{F}\cdot d\vec{r}$

(2.25 puntu)



$$S\equiv z=x^2+y^2 \quad \forall(x,y)\in R_{xy}$$

$$\text{non } R_{xy}\equiv\begin{cases}x^2+y^2\le 1\\ x\ge 0 \quad y\ge 0\end{cases}$$

$$\begin{cases}\vec{N}=(-2x,-2y,1)\\ \gamma>\dfrac{\pi}{2}\end{cases}$$

S-ren muga, marrazkian urdinez adierazitako *C* kurba itxia da:

$$C=C_1\cup C_2\cup C_3$$

a) $$\iint_S \vec{F}\cdot d\vec{S}=\iint_S\left(ydydz+zdzdx+xdxdy\right)=\pm\iint_{R_{xy}}\left(\vec{F}\cdot\vec{N}\right)dxdy=$$

$$=-\iint_{R_{xy}}\left(-2xy-2y\left(x^2+y^2\right)+x\right)dxdy=$$

$$\text{Polarretan }\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases} \quad |J|=\rho \quad\Rightarrow\quad R_{xy}\equiv R_{\theta\rho}\begin{cases}0\le\rho\le 1\\ 0\le\theta\le\dfrac{\pi}{2}\end{cases}$$

$$=\int_0^{\frac{\pi}{2}}\int_0^1\rho\left(2\rho^2\cos\theta\sin\theta+2\rho^3\sin\theta-\rho\cos\theta\right)d\rho\, d\theta=\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\cos\theta\sin\theta+\frac{2}{5}\sin\theta-\frac{1}{3}\cos\theta\right)d\theta=$$

$$=\left[\frac{\sin^2\theta}{4}-\frac{2}{5}\cos\theta-\frac{1}{3}\sin\theta\right]_0^{\frac{\pi}{2}}=\frac{1}{4}-\frac{1}{3}+\frac{2}{5}=\frac{19}{60}$$

<u>Beste era batera</u>:

$\vec{F}$ eta bere lehenengo deribatu partzialak jarraituak dira $D=\mathbb{R}^3$ eremuan, eta $div\left(\vec{F}\right)=0$ denez, Gauss-en teorema erabiltzea erabaki genezake. Baina *S* ez da gainazal itxia beraz, itxi beharko dugu. Eta, horretarako, 3 “estalki” jarri behar zaizkio:

$$\left.\begin{array}{l} S_1 \equiv z=1 \quad \forall(x,y)\in R_{xy} \equiv \begin{cases} x^2+y^2\le 1 \\ x\ge 0 \quad y\ge 0 \end{cases} \\ S_2 \equiv x=0 \quad \forall(y,z)\in R_{yz} \equiv \begin{cases} 0\le y\le 1 \\ y^2\le z\le 1 \end{cases} \\ S_3 \equiv y=0 \quad \forall(y,z)\in R_{xz} \equiv \begin{cases} 0\le x\le 1 \\ x^2\le z\le 1 \end{cases} \end{array}\right\} \Rightarrow S' = S\cup S_1\cup S_2\cup S_3 \text{ gainazal itxia da.}$$

Orduan:

$$\iint_{S'} \vec{F}\cdot d\vec{S} = \iiint_V div\left(\vec{F}\right)dxdydz = 0 = \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S}$$

Hiru integral kalkulatu behar ditugu:

$$\iint_{S_1} \vec{F}\cdot d\vec{S} \overset{(1)}{=} \iint_{R_{xy}} x\,dxdy = \int_0^{\pi/2}\int_0^1 \rho^2\cos\theta d\rho d\theta = \frac{1}{3}$$

(1) $\begin{cases} \vec{N}=(0,0,1) \\ \gamma < \dfrac{\pi}{2} \end{cases}$. Polarretan, $R_{xy} \equiv R_{\theta\rho}\begin{cases} 0\le \rho\le 1 \\ 0\le\theta\le\dfrac{\pi}{2} \end{cases}$

$$\iint_{S_2} \vec{F}\cdot d\vec{S} \overset{(2)}{=} -\iint_{R_{yz}} y\,dydz = -\int_0^1\int_{y^2}^1 ydzdy = -\int_0^1 y\left(1-y^2\right)dy = -\frac{1}{4}$$

(2) $\begin{cases} \vec{N}=(1,0,0) \\ \alpha > \dfrac{\pi}{2} \end{cases}$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} \overset{(3)}{=} -\iint_{R_{xz}} z\,dzdx = -\int_0^1\int_{x^2}^1 zdzdx = -\frac{1}{2}\int_0^1\left(1-x^4\right)dx = -\frac{2}{5}$$

(3) $\begin{cases} \vec{N}=(0,1,0) \\ \beta > \dfrac{\pi}{2} \end{cases}$

Eta, azkenik, $\iint_S \vec{F}\cdot d\vec{S} = -\iint_{S_1} \vec{F}\cdot d\vec{S} - \iint_{S_2} \vec{F}\cdot d\vec{S} - \iint_{S_3} \vec{F}\cdot d\vec{S} = \dfrac{19}{60}$

b) $\oint_C \vec{F}\cdot d\vec{r} = \oint_C \left(ydx+zdy+xdz\right) = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$

non $C_1 \equiv \begin{cases} x^2+y^2=1 \\ z=1 \end{cases} \equiv \begin{cases} x=\cos t \\ y=\sin t \\ z=1 \end{cases}$ t $\dfrac{\pi}{2}$-tik 0-ra $\Rightarrow$

$$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{\pi/2}^{0} \left(-\sin^2 t+\cos t\right)dt = \int_{\pi/2}^{0}\left(\frac{\cos(2t)-1}{2}+\cos t\right)dt =$$

$$=\left[\frac{\sin(2t)}{4}-\frac{t}{2}+\sin t\right]_{\pi/2}^{0} = \frac{\pi}{4}-1$$

$$C_2 \equiv \begin{cases} z = x^2 \\ y = 0 \end{cases} \quad x \text{ 1-etik 0-ra} \quad \Rightarrow \int_{C_2} \vec{F}\cdot d\vec{r} = \int_1^0 x\cdot 2x dx = \int_1^0 2x^2 dx = -\frac{2}{3}$$

$$C_3 \equiv \begin{cases} z = y^2 \\ x = 0 \end{cases} \quad 0 \le y \le 1 \quad \Rightarrow \int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^1 y^2 dy = \frac{1}{3}$$

Beraz, $\oint_C \vec{F}\cdot d\vec{r} = \frac{\pi}{4}-1-\frac{2}{3}+\frac{1}{3} = \frac{\pi}{4}-\frac{4}{3}$

<u>Beste era batera</u>:

$\vec{F}$ eta bere lehenengo deribatu partzialak jarraituak dira $D=\mathbb{R}^3$ eremuan, eta C kurba itxia beraz, Stokes erabiliz:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \overrightarrow{rot}\left(\vec{F}\right)\cdot dS = \pm\iint_{R_{xy}}\left(\overrightarrow{rot}\left(\vec{F}\right)\cdot\vec{N}\right)dxdy =$$

non $S \equiv z = x^2+y^2 \quad \forall (x,y)\in R_{xy} \equiv \begin{cases} x^2+y^2 \le 1 \\ x\ge 0 \quad y \ge 0 \end{cases} \Rightarrow \begin{cases} \vec{N} = (-2x,-2y,1) \\ \gamma > \frac{\pi}{2} \end{cases}$

eta $\overrightarrow{rot}\left(\vec{F}\right) = (-1,-1,-1)$

Polarretan $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \quad |J| = \rho \quad \Rightarrow \quad R_{xy} \equiv R_{\theta\rho}\begin{cases} 0\le\rho\le 1 \\ 0\le\theta\le\frac{\pi}{2} \end{cases}$:

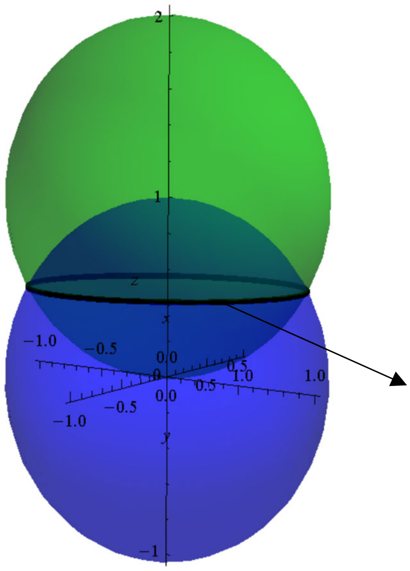
$$=-\iint_{R_{xy}}\left(2x+2y-1\right)dxdy = -\int_0^{\frac{\pi}{2}}\int_0^1 \rho\left(2\rho(\cos\theta+\sin\theta)-1\right)d\rho\, d\theta = -\int_0^{\frac{\pi}{2}}\left(\frac{2}{3}(\cos\theta+\sin\theta)-\frac{1}{2}\right)d\theta =$$

$$=-\left(\frac{2}{3}(\sin\theta-\cos\theta)-\frac{t}{2}\right)\Bigg|_0^{\pi/2} = \frac{\pi}{4}-\frac{4}{3}$$

5.- a) Kalkulatu V solidoa mugatzen duen $S=S_1\cup S_2$ gainazal itxian zehar irteten den $\vec{F}(x,y,z)=(y+z)\vec{i}+y\vec{j}+z\vec{k}$ eremu bektorialaren fluxua, non $\begin{cases} S_1\equiv x^2+y^2+z^2=1 & (z\geq 0) \\ S_2\equiv x^2+y^2+z^2-2z=0 & (z\leq 1)\end{cases}$.

b) Kalkulatu S gainazala osatzen duen S_1 gainazalaren zatiaren azalera.

(2.25 puntu)



a) S gainazal itxia denez, eta $\vec{F}$ eta bere lehenengo deribatu partzialak jarraituak direnez, Gauss-en teorema erabil daiteke:

$$\iint_S \vec{F}\cdot d\vec{S}=\iiint_V div\left(\vec{F}\right)dxdydz=\iiint_V 2\ dxdydz$$

Zilindrikoetan: $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \quad |J|=\rho \\ z=z\end{cases}$

$$S_1\cap S_2=C\equiv\begin{cases} x^2+y^2=3/4 \\ z=1/2\end{cases}\equiv\begin{cases}\rho=\dfrac{\sqrt{3}}{2} \\ z=1/2\end{cases}$$

$$\begin{cases} S_1\equiv \rho^2+z^2=1 & (z\geq 0) \\ S_2\equiv \rho^2+(z-1)^2=1 & (z\leq 1)\end{cases}\Leftrightarrow\begin{cases} S_1\equiv z=\sqrt{1-\rho^2} \\ S_2\equiv z=1-\sqrt{1-\rho^2}\end{cases}$$

Beraz, $V\equiv\begin{cases} 0\leq\theta\leq 2\pi \\ 0\leq\rho\leq\dfrac{\sqrt{3}}{2} \\ 1-\sqrt{1-\rho^2}\leq z\leq\sqrt{1-\rho^2}\end{cases}$, eta:

$$\iint_S \vec{F}\cdot d\vec{S}=2\iiint_V dxdydz=2\int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{1-\sqrt{1-\rho^2}}^{\sqrt{1-\rho^2}}\rho\,dz\,d\rho\,d\theta=4\pi\int_0^{\frac{\sqrt{3}}{2}}\rho\left(\sqrt{1-\rho^2}-1+\sqrt{1-\rho^2}\right)d\rho=$$

$$=4\pi\int_0^{\frac{\sqrt{3}}{2}}\rho\left(2\sqrt{1-\rho^2}-1\right)d\rho=4\pi\left(-\frac{\left(1-\rho^2\right)^{3/2}}{3/2}-\frac{\rho^2}{2}\right)_0^{\frac{\sqrt{3}}{2}}=4\pi\left(-\frac{1}{12}-\frac{3}{8}+\frac{2}{3}\right)=\frac{5\pi}{6}$$

b) $Azalera\left(S_1\right)=\iint_{R_{xy}}\left|\vec{N}\right|dxdy\overset{(1)}{=}\iint_{R_{xy}}\dfrac{1}{\sqrt{1-x^2-y^2}}dxdy\overset{(2)}{=}\int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\dfrac{\rho}{\sqrt{1-\rho^2}}d\rho\,d\theta=$

$$= 2\pi\left(-\sqrt{1-\rho^2}\right)_0^{\frac{\sqrt{3}}{2}} = 2\pi\left(-\frac{1}{2}+1\right) = \pi$$

(1) $S_1 \equiv z = \sqrt{1-x^2-y^2} \quad \forall (x,y) \in R_{xy} \equiv x^2+y^2 \le 3/4$

$$\vec{N} = \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right) \Rightarrow \left|\vec{N}\right| = \frac{1}{\sqrt{1-x^2-y^2}}$$

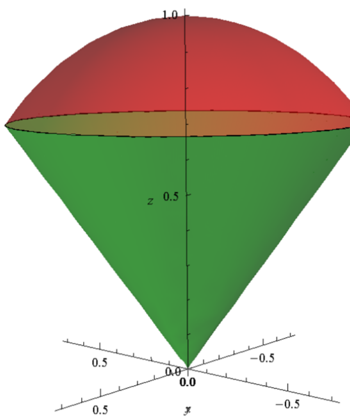
(2) Polarretan $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad |J| = \rho \quad \Rightarrow \quad R_{xy} \equiv \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \rho \le \frac{\sqrt{3}}{2} \end{cases}$

6.- Kalkulatu $V \equiv \begin{cases} x^2+y^2+z^2 \le 1 \\ z \ge \sqrt{x^2+y^2} \end{cases}$ **solidoaren masa-zentroa, dentsitatea konstantea dela jakinda.**

OHARRA: Dentsitate konstantea duen V **solidoaren masa-zentroaren koordenatuak** $(\bar{x},\bar{y},\bar{z})$ **dira, non:**

$$\bar{x}=\frac{\iiint_V x\,dxdydz}{\text{Bolumena}(V)} \quad \bar{y}=\frac{\iiint_V y\,dxdydz}{\text{Bolumena}(V)} \quad \bar{z}=\frac{\iiint_V z\,dxdydz}{\text{Bolumena}(V)}$$

(1.25 puntu)



V solidoa simetrikoa da $x=0$ eta $y=0$ planoekiko. Eta, x eta y funtzio bakoitiak dira, beraz:

$$\iiint_V x\,dxdydz=0 \text{ eta } \iiint_V y\,dxdydz=0$$

Orain, $\iiint_V z\,dxdydz$ eta V-ren bolumena ematen duen $\iiint_V dxdydz$ integrala kalkulatzeko, koordenatu esferikoak erabiliko ditugu:

$$\begin{cases} x=\rho\cos\theta\sin\varphi \\ y=\rho\sin\theta\sin\varphi \\ z=\rho\cos\varphi \end{cases} \quad |J|=\rho^2\sin\varphi \quad \Rightarrow \quad V\equiv\begin{cases}\rho^2\le 1 \\ \cos\varphi\ge\sin\varphi\end{cases} \equiv \begin{cases} 0\le\theta\le 2\pi \\ 0\le\rho\le 1 \\ 0\le\varphi\le\pi/4\end{cases}$$

Orduan:

$$\iiint_V z\,dxdydz=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3\sin\varphi\cos\varphi\,d\rho\,d\varphi\,d\theta=2\pi\int_0^{\pi/4}\frac{\sin\varphi\cos\varphi}{4}d\varphi=\frac{\pi}{2}\frac{\sin^2\varphi}{2}\Big|_0^{\pi/4}=\frac{\pi}{8}$$

Eta,

$$\iiint_V dxdydz=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta=2\pi\int_0^{\pi/4}\frac{\sin\varphi}{3}d\varphi=\frac{2\pi}{3}(-\cos\varphi)\Big|_0^{\pi/4}=\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

Beraz, $\bar{z}=\dfrac{\iiint_V z\,dxdydz}{\text{Bolumena}(V)}=\dfrac{\frac{\pi}{8}}{\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)}=\dfrac{3}{8\left(2-\sqrt{2}\right)}$

Eta masa-zentroa $(\bar{x},\bar{y},\bar{z})=\left(0,0,\dfrac{3}{8\left(2-\sqrt{2}\right)}\right)$

Esferikoetan beharrean, zilindrikoetan planteatzen badugu:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \quad |J| = \rho \quad \Rightarrow \quad V \equiv \begin{cases} \rho^2 + z^2 \le 1 \\ z \ge \rho \end{cases} \overset{\text{ebakidura}}{\Rightarrow} \quad 2\rho^2 = 1 \quad \Leftrightarrow \quad z = \rho = \frac{1}{\sqrt{2}}$$

Orduan, $V \equiv \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \rho \le \dfrac{1}{\sqrt{2}} \\ \rho \le z \le \sqrt{1-\rho^2} \end{cases}$, eta:

$$\iiint_V dxdydz = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_\rho^{\sqrt{1-\rho^2}} \rho\, dz\, d\rho\, d\theta = 2\pi \int_0^{1/\sqrt{2}} \rho\left(\sqrt{1-\rho^2} - \rho\right) d\rho =$$

$$= 2\pi\left[-\frac{\left(1-\rho^2\right)^{3/2}}{3} - \frac{\rho^3}{3}\right]_0^{\frac{1}{\sqrt{2}}} = -\frac{2\pi}{3}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - 1\right) = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$\iiint_V zdxdydz = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_\rho^{\sqrt{1-\rho^2}} \rho z\, dz\, d\rho\, d\theta = \pi \int_0^{1/\sqrt{2}} \rho\left(1 - \rho^2 - \rho^2\right) d\rho = \pi\left[\frac{\rho^2}{2} - \frac{\rho^4}{4}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{8}$$