Ariketa 1	Ariketa 2	Ariketa 3	Ariketa 4	1. zatia

Azterketaren iraupena: 3 ordu

IZEN-ABIZENAK:

TALDEA:

1.- a) Aztertu $\sum_{n=1}^{\infty}\left(\frac{an+1}{n}\right)^n$ **seriearen izaera** $\forall a>0$.

b) Kalkulatu $\lim_{n\to\infty}\dfrac{\frac{a+1}{1}+\left(\frac{2a+1}{2}\right)^2+\ldots+\left(\frac{an+1}{n}\right)^n}{1+\frac{1}{2}+\ldots+\frac{1}{n}}$ **non** $a>0$.

(2 puntu)

a) $\sum_{n=1}^{\infty}a_n$ non $a_n=\left(\frac{an+1}{n}\right)^n\geq 0 \quad \forall n\in\mathbb{N}$ eta $\forall a>0$.

Cauchy-ren irizpidea erabiliz:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{an+1}{n}=a\begin{cases}<1 \;\Rightarrow\; \sum_{n=1}^{\infty}a_n \text{ konbergentea da}\\ >1 \;\Rightarrow\; \sum_{n=1}^{\infty}a_n \text{ dibergentea da}\\ =1 \;\Rightarrow\; \text{kasu zalantzakoa} \;\Rightarrow\; a_n=\left(\frac{n+1}{n}\right)^n \;\Rightarrow\end{cases}$$

$\Rightarrow \lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\neq 0 \;\Rightarrow$ Baldintza beharrezkoa ez da betetzen $\Rightarrow \sum_{n=1}^{\infty}a_n$ dibergentea da.

$$\text{b)}\ \lim_{n\to\infty}\frac{\frac{a+1}{1}+\left(\frac{2a+1}{2}\right)^2+\ldots+\left(\frac{an+1}{n}\right)^n}{1+\frac{1}{2}+\ldots+\frac{1}{n}}=\frac{\sum_{n=1}^{\infty}\left(\frac{an+1}{n}\right)^n}{\sum_{n=1}^{\infty}\frac{1}{n}}\overset{(*)}{=}\begin{cases}\frac{S\in\mathbb{R}^+}{\infty}=0 & \forall a<1\\ \frac{\infty}{\infty} & \forall a\geq 1 \overset{(**)}{\Rightarrow}\end{cases}$$

$$\overset{(**)}{\Rightarrow} \lim_{n\to\infty} \frac{\frac{a+1}{1}+\left(\frac{2a+1}{2}\right)^2+\ldots+\left(\frac{an+1}{n}\right)^n}{1+\frac{1}{2}+\ldots+\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(\frac{an+1}{n}\right)^n}{\frac{1}{n}} = \begin{cases} \frac{a^\infty}{0} = \frac{\infty}{0} = \infty & \forall a > 1 \\ \frac{e}{0} = \infty & \text{baldin } a = 1 \end{cases}$$

(*) a) atalean frogatu dugunez:

$$\sum_{n=1}^{\infty}\left(\frac{an+1}{n}\right)^n \begin{cases} \text{konbergente da } \forall a < 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty}\left(\frac{an+1}{n}\right)^n = S \in \mathbb{R} \\ \text{dibergente da } \forall a \geq 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty}\left(\frac{an+1}{n}\right)^n = \infty \end{cases}$$

(**) $\left\{1+\frac{1}{2}+\ldots+\frac{1}{n}\right\}$ hertsiki gorakorra eta dibergentea da beraz, Stolz erabil daiteke.

2.- Aztertu $\sum_{n=1}^{\infty}\left[\frac{1}{7}\cdot\frac{2^{n-1}}{3^n}-\mathrm{L}\left(1+\frac{1}{7n^2}\right)\right]$ **seriearen izaera.**

(2 puntu)

$$\sum_{n=1}^{\infty}\left[\frac{1}{7}\cdot\frac{2^{n-1}}{3^n}-\mathrm{L}\left(1+\frac{1}{7n^2}\right)\right]=\sum_{n=1}^{\infty}\left[a_n-b_n\right]$$

non $a_n=\frac{1}{7}\cdot\frac{2^{n-1}}{3^n}\geq 0$ eta $b_n=\mathrm{L}\left(1+\frac{1}{7n^2}\right)\geq 0 \quad \forall n\in\mathbb{N}$

- $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{7}\cdot\frac{2^{n-1}}{3^n}=\sum_{n=1}^{\infty}\frac{1}{21}\cdot\left(\frac{2}{3}\right)^{n-1} \Rightarrow$ serie geometrikoa da.

  Arrazoia $r=\frac{2}{3}<1 \Rightarrow \sum_{n=1}^{\infty}a_n$ konbergentea da.

- $b_n=\mathrm{L}\left(1+\frac{1}{7n^2}\right)\sim\frac{1}{7n^2} \Rightarrow \sum_{n=1}^{\infty}b_n$ konbergentea da.

Orduan:

$$\sum_{n=1}^{\infty}\left[\frac{1}{7}\cdot\frac{2^{n-1}}{3^n}-\mathrm{L}\left(1+\frac{1}{7n^2}\right)\right]=\sum_{n=1}^{\infty}\left[a_n-b_n\right]=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n \text{ konbergentea da.}$$

3.- Aurkitu $f(x)=\mathrm{L}\left(1+x^3\right)$ **funtzioaren berretura-seriezko garapena, non balio duen adieraziz.**

(2 puntu)

$$f(x)=\mathrm{L}\left(1+x^3\right) \quad \Rightarrow \quad f'(x)=\frac{3x^2}{1+x^3} \overset{(*)}{=} \sum_{n=0}^{\infty} 3x^2\cdot\left(-x^3\right)^n = \sum_{n=0}^{\infty} 3\cdot(-1)^n\cdot x^{3n+2} \quad \forall x\in(-1,1)$$

(*) $r=-x^3$ arrazoiko serie geometrikoaren batura, konbergentea $\Leftrightarrow |r|=\left|-x^3\right|=\left|x^3\right|<1$

Eta integragarria da $\left[0,x\right]$ tartean $\forall x\in(-1,1)$:

$$f(x)-\underbrace{f(0)}_{=0}=\sum_{n=0}^{\infty} 3\cdot(-1)^n\cdot\frac{x^{3n+3}}{3n+3}=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{3n+3}}{n+1}=S(x) \quad \forall x\in(-1,1)$$

$x=-1$ puntuan $\nexists f$

$$x=1 \text{ puntuan } \begin{cases} \exists f \text{ jarraitua} \\ \exists \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} \text{ baldintzaz konbergentea} \quad \Rightarrow \quad \exists S \text{ jarraitua} \\ f(x)=S(x) \quad \forall x\in(-1,1) \end{cases}$$

$$\Rightarrow \quad f(x)=\mathrm{L}\left(1+x^3\right)=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{3n+3}}{n+1} \quad \forall x\in\left(-1,1\right]$$

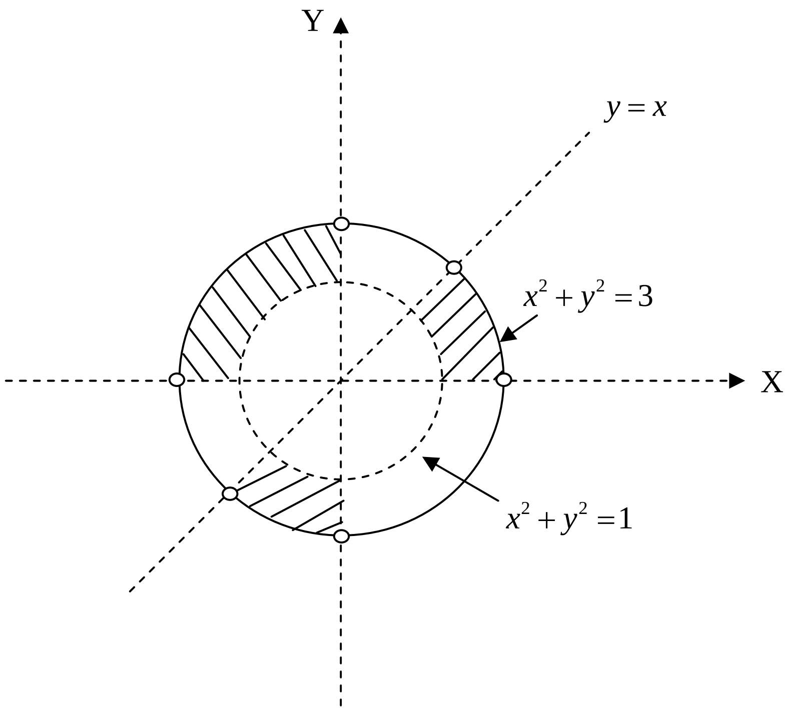
4.- Aurkitu analitiko eta grafikoki hurrengo funtzioaren definizio-eremua:

$$f(x,y)=\mathrm{L}\left(\frac{xy}{x-y}\right)+\frac{\sqrt{3-x^2-y^2}}{\sqrt{x^2+y^2-1}}$$

(2 puntu)

$$D=\left\{(x,y)\in\mathbb{R}^2 \,/\, \frac{xy}{x-y}>0,\, x-y\neq 0,\, 3-x^2-y^2\geq 0,\, x^2+y^2-1>0\right\}$$

- $$\frac{xy}{x-y}>0 \;\Leftrightarrow\; \begin{cases} x-y>0 \quad \wedge \quad xy>0 \\ \qquad\qquad \vee \\ x-y<0 \quad \wedge \quad xy<0 \end{cases} \;\Leftrightarrow\; \begin{cases} x>y \quad \wedge \quad \begin{cases} x>0 \wedge y>0 \\ \qquad \vee \\ x<0 \wedge y<0 \end{cases} \\ \vee \\ x<y \quad \wedge \quad \begin{cases} x>0 \wedge y<0 \;\# \\ \vee \\ x<0 \wedge y>0 \end{cases} \end{cases}$$
- $x-y\neq 0 \quad\Leftrightarrow\quad x\neq y$
- $3-x^2-y^2\geq 0 \quad\Leftrightarrow\quad x^2+y^2\leq 3$
- $x^2+y^2-1>0 \quad\Leftrightarrow\quad x^2+y^2>1$



Ariketa 5	Ariketa 6	Ariketa 7	2. zatia

Azterketaren iraupena: 3 ordu

IZEN-ABIZENAK:

TALDEA:

5.- $f(x,y)=\begin{cases} x+e^{\frac{x^2\cdot y^3}{x^2+y^2}} & \forall (x,y)\neq(0,0) \\ 1 & (x,y)=(0,0)\end{cases}$ **funtzioa emanik,**

a) Aztertu bere jarraitutasuna (0,0) puntuan.
b) Kalkulatu bere deribatu partzialak (0,0) puntuan.
c) Aztertu bere diferentziagarritasuna (0,0) puntuan.
d) Aztertu bere deribagarritasuna (0,0) puntuan eta kalkulatu puntu horretan bere deribatu direkzionala $\vec{u}=(1,1)$ bektorearen norabidean.

(3 puntu)

a) $$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\left(x+e^{\frac{x^2\cdot y^3}{x^2+y^2}}\right)=0+\lim_{(x,y)\to(0,0)} e^{\frac{x^2\cdot y^3}{x^2+y^2}} \overset{(*)}{=} \lim_{\substack{\rho\to 0^+\\ \forall\theta\in[0,2\pi)}} e^{\frac{\rho^5\cdot\cos^2\theta\cdot\sin^3\theta}{\rho^2}}=$$

$$=\lim_{\substack{\rho\to 0^+\\ \forall\theta\in[0,2\pi)}} e^{\rho^3\cdot\cos^2\theta\cdot\sin^3\theta}=e^0=1=f(0,0) \quad \Leftrightarrow \quad f \text{ jarraitua da (0,0) puntuan.}$$

b) $$f_x'(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h+e^{\frac{0}{h^2}}-1}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

$$f_y'(0,0)=\lim_{hk\to 0}\frac{f(0,k)-f(0,0)}{h}=\lim_{k\to 0}\frac{e^{\frac{0}{k^2}}-1}{k}=\lim_{k\to 0}\frac{0}{k}=0$$

c) Baldintza beharrezkoa eta nahikoa aplikatuz:

$$\lim_{(h,k)\to(0,0)}\frac{\left|f(h,k)-f(0,0)-h\cdot f_x'(0,0)-k\cdot f_y'(0,0)\right|}{\sqrt{h^2+k^2}}=$$

$$=\lim_{(h,k)\to(0,0)}\frac{\left|h+e^{\frac{h^2\cdot k^3}{h^2+k^2}}-1-h\right|}{\sqrt{h^2+k^2}}=\lim_{(h,k)\to(0,0)}\frac{\left|e^{\frac{h^2\cdot k^3}{h^2+k^2}}-1\right|}{\sqrt{h^2+k^2}}\overset{(*)}{=}\lim_{\substack{\rho\to 0^+\\ \forall\theta\in[0,2\pi)}}\frac{\left|e^{\rho^3\cdot\cos^2\theta\cdot\sin^3\theta}-1\right|}{\rho}\sim$$

$$\sim \lim_{\substack{\rho\to 0^+ \\ \forall\theta\in[0,2\pi)}} \frac{\rho^3\cdot\cos^2\theta\cdot\left|\sin^3\theta\right|}{\rho} = \lim_{\substack{\rho\to 0^+ \\ \forall\theta\in[0,2\pi)}} \rho^2\cdot\cos^2\theta\cdot\left|\sin^3\theta\right| = 0 \quad \Leftrightarrow \quad f \text{ diferentziagarria da } (0,0) \text{ puntuan}$$

d) f diferentziagarria da (0,0) puntuan $\Rightarrow f$ deribagarria da (0,0) puntuan eta

$$\left.\frac{df}{d\vec{u}}\right|_{(0,0)} = f'_x(0,0)\cdot h_1 + f'_y(0,0)\cdot h_2 \quad \forall\vec{u} = \left(h_1, h_2\right) \text{ unitario}$$

Kasu honetan, $\left|\vec{u}\right| = \sqrt{2} \quad \Rightarrow \quad \vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ unitarioa da eta $\left.\frac{df}{d\vec{u}}\right|_{(0,0)} = \frac{1}{\sqrt{2}}$.

6.- Izan bedi *f* funtzio diferentziagarria *P* puntuan.

a) Adierazi, arrazoituz, hurrengo hiru kasuetan zein den $\vec{u}$ bektoreak adierazten duen norabidea:

i. $\left.\frac{df}{d\vec{u}}\right|_P = 0$

ii. $\left.\frac{df}{d\vec{u}}\right|_P = f'_x(P)$

iii. $\left.\frac{df}{d\vec{u}}\right|_P = \left|\overrightarrow{\nabla f}(P)\right|$

b) Izan bedi $\overrightarrow{\nabla f}(P) = (3,-4)$. $\exists\,\vec{u}$ bektore unitarioa zeinerako $\left.\frac{df}{d\vec{u}}\right|_P = 6$?

Erantzuna arrazoitu.

(2 puntu)

a) f diferentziagarria denez $\Rightarrow \left.\frac{df}{d\vec{u}}\right|_P = \overrightarrow{\nabla f}(P)\cdot\vec{u} = f'_x(P)\cdot h_1 + f'_y(P)\cdot h_2$ non $\vec{u} = (h_1, h_2)$ unitarioa den. Hori dela eta:

i. $\left.\frac{df}{d\vec{u}}\right|_P = 0 \iff \vec{u} \perp \overrightarrow{\nabla f}(P) \iff \vec{u}$ maila-kurbaren norabidea da.

ii. $\left.\frac{df}{d\vec{u}}\right|_P = f'_x(P) \iff \vec{u}\quad OX$ ardatzaren norabidea da.

iii. $\left.\frac{df}{d\vec{u}}\right|_P = \left|\overrightarrow{\nabla f}(P)\right| \iff \vec{u}\quad \overrightarrow{\nabla f}(P)$ -ren norabidea da.

b) $\overrightarrow{\nabla f}(P) = (3,-4) \Rightarrow \left.\frac{df}{d\vec{u}}\right|_P = \left|\overrightarrow{\nabla f}(P)\right| = 5$ deribatu direkzional maximoa da

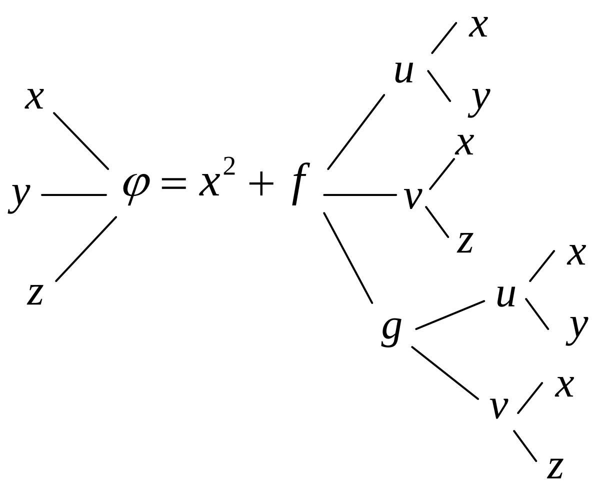
$\Rightarrow \nexists\,\vec{u}$ unitario non $\left.\frac{df}{d\vec{u}}\right|_P = 6 > 5$

7.- Izan bedi $\varphi(x,y,z)=x^2+f(xy,-xz,g(xz,xy))$ **non** φ**,** f **eta** g **funtzio diferentziagarriak diren. Kalkulatu hurrengo adierazpenaren balioa:**

$$E\equiv\varphi'_x(1,1,1)-\varphi'_y(1,1,1)-\varphi'_z(1,1,1)$$

(2 puntu)

$$\varphi(x,y,z)=x^2+f(u,-v,g(v,u))\ \text{non}\ \begin{cases}u=xy\\ v=xz\end{cases}$$



$$\left.\begin{aligned}
&\varphi'_x=2x+f'_u\cdot u'_x-f'_v\cdot v'_x+f'_g\cdot\left(g'_v\cdot v'_x+g'_u\cdot u'_x\right)=2x+y\cdot f'_u-z\cdot f'_v+f'_g\cdot\left(z\cdot g'_v+y\cdot g'_u\right)\\
&\varphi'_y=f'_u\cdot u'_y+f'_g\cdot g'_u\cdot u'_y=x\cdot f'_u+f'_g\cdot x\cdot g'_u\\
&\varphi'_z=-f'_v\cdot v'_z+f'_g\cdot g'_v\cdot v'_z=-x\cdot f'_v+f'_g\cdot x\cdot g'_v
\end{aligned}\right\}\Rightarrow$$

$$\left.\begin{aligned}
&\varphi'_x(1,1,1)=2+f'_u(1,-1,g(1,1))-f'_v(1,-1,g(1,1))+f'_g(1,-1,g(1,1))\cdot\left(g'_v(1,1)+g'_u(1,1)\right)\\
&\varphi'_y(1,1,1)=f'_u(1,-1,g(1,1))+f'_g(1,-1,g(1,1))\cdot g'_u(1,1)\\
&\varphi'_z(1,1,1)=-f'_v(1,-1,g(1,1))+f'_g(1,-1,g(1,1))\cdot g'_v(1,1)
\end{aligned}\right\}\Rightarrow$$

$$\begin{aligned}
E&\equiv\varphi'_x(1,1,1)-\varphi'_y(1,1,1)-\varphi'_z(1,1,1)=\\
&=2+f'_u(1,-1,g(1,1))-f'_v(1,-1,g(1,1))+f'_g(1,-1,g(1,1))\cdot\left(g'_v(1,1)+g'_u(1,1)\right)\\
&\quad-f'_u(1,-1,g(1,1))-f'_g(1,-1,g(1,1))\cdot g'_u(1,1)\\
&\quad+f'_v(1,-1,g(1,1))-f'_g(1,-1,g(1,1))\cdot g'_v(1,1)=2
\end{aligned}$$