Azterketaren iraupena: 2 ordu eta erdi

OHARRA: Azterketako emaitza guztiak behar den bezala arrazoitu behar dira.

1.- Kalkulatu $\lim_{n\to\infty}\frac{n^2+(n+1)^2+\ldots+(2n)^2}{n^3}$

(0.75 puntu)

$\{n^3\}$ hertsiki gorakorra eta dibergentea da, beraz, Stolz erabil daiteke:

$$\lim_{n\to\infty}\frac{n^2+(n+1)^2+\ldots+(2n)^2}{n^3}=\lim_{n\to\infty}\frac{n^2+(n+1)^2+\ldots+(2n)^2-(n-1)^2-n^2-\ldots-(2n-2)^2}{n^3-(n-1)^3}=$$

$$=\lim_{n\to\infty}\frac{(2n)^2+(2n-1)^2-(n-1)^2}{n^3-(n-1)^3}=\lim_{n\to\infty}\frac{4n^2+(4n^2-4n+1)-(n^2-2n+1)}{n^3-(n^3-3n^2+3n-1)}=\lim_{n\to\infty}\frac{7n^2}{3n^2}=\frac{7}{3}$$

2.- $a_n=\frac{2^n}{n^3+b^n}\quad \forall b\in\mathbb{R}^+$ **, gai orokorra emanik,**

a) Kalkulatu $\lim_{n\to\infty}a_n$

b) Aztertu $\sum_{n=1}^{\infty}a_n$ **seriearen izaera**

(2 puntu)

a) $\forall b\in\mathbb{R}^+\quad \lim_{n\to\infty}b^n=\begin{cases}0 & \forall b<1\\ 1 & b=1\\ \infty & \forall b>1\end{cases}\;\Rightarrow\; \lim_{n\to\infty}\frac{2^n}{n^3+b^n}=\begin{cases}\lim_{n\to\infty}\frac{2^n}{n^3}=\infty & \forall b<1\\ \lim_{n\to\infty}\frac{2^n}{n^3+1}=\infty & b=1\\ \lim_{n\to\infty}\frac{2^n}{b^n}=\lim_{n\to\infty}\left(\frac{2}{b}\right)^n & \forall b>1\end{cases}$

Eta, $\forall b>1\quad \lim_{n\to\infty}\frac{2^n}{n^3+b^n}=\lim_{n\to\infty}\left(\frac{2}{b}\right)^n=\begin{cases}\infty & \forall b<2\\ 1 & b=2\\ 0 & \forall b>2\end{cases}$

b) $\forall b\le 2\quad \lim_{n\to\infty}a_n\neq 0\quad\Rightarrow\quad\sum_{n=1}^{\infty}a_n$ dibergentea da (konbergentea izateko bete behar den BB ez da egiaztatzen, eta $a_n\ge 0\quad\forall n$).

$\forall b>2\quad \lim_{n\to\infty}a_n=0\quad\Rightarrow\quad\sum_{n=1}^{\infty}a_n$ konbergentea izan daiteke (BB betetzen da). Eta, kasu honetan, $a_n=\frac{2^n}{n^3+b^n}\sim\left(\frac{2}{b}\right)^n$, non $\sum_{n=1}^{\infty}\left(\frac{2}{b}\right)^n$ serie geometrikoa da, $|r|=\frac{2}{b}<1$, beraz konbergentea. Orduan, $\sum_{n=1}^{\infty}a_n$ konbergentea da.

3.- Aztertu $\sum_{n=1}^{\infty}\left(\left(\frac{n+1}{n}\right)^n-\frac{2n}{n+1}\right)^{-n}$ **seriearen izaera**

(0.5 puntu)

$$\sum_{n=1}^{\infty}a_n \text{ non } a_n=\left(\left(\frac{n+1}{n}\right)^n-\frac{2n}{n+1}\right)^{-n}=\frac{1}{\left(\left(\frac{n+1}{n}\right)^n-\frac{2n}{n+1}\right)^n}>0 \quad \forall n$$

Konbergentzi baldintza beharrezkoa erabiliz:

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{\left(\left(\frac{n+1}{n}\right)^n-\frac{2n}{n+1}\right)^n}=\frac{1}{(e-2)^{\infty}}\overset{(1)}{=}\frac{1}{0}=\infty\neq 0 \quad\Rightarrow\quad \sum_{n=1}^{\infty}a_n \text{ dibergentea da}$$

(1) $0<e-2<1$

4.- $f(x)=e^x$ **funtzioaren berretura-seriezko garapena erabiliz,** $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \forall x\in\mathbb{R}$, **determinatu, arrazoituz, zenbat batugai hartu behar dira garapen horretan** $e^{-1/4}$ **-aren balio hurbildua kalkulatzeko, errorea** 10^{-2} **baino txikiagoa izanik.**

(Puntu 1)

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \forall x\in\mathbb{R} \quad\Rightarrow\quad e^{-1/4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n\cdot n!}$$

Serie alternatua da, eta, Leibniz-en teorema egiaztatzen du:

$$a_n=\frac{(-1)^n}{4^n\cdot n!} \quad\Rightarrow\quad \begin{cases}\lim_{n\to\infty}|a_n|=0\\ |a_n|>|a_{n+1}|\end{cases}$$

Eta, teorema horren ondorioz:

$$\text{Errorea}=|S-S_n|<|a_{n+1}|, \qquad \text{non } S=e^{-1/4} \text{ eta } S_n \text{ balio hurbildua}$$

Ariketa honetan *n* da kalkulatu behar duguna. Beraz:

$$|S-S_n|<\frac{1}{4^{n+1}\cdot(n+1)!}\leq\frac{1}{100} \quad\Leftrightarrow\quad 4^{n+1}\cdot(n+1)!\geq 100$$

Eta, *n*-ri balioak emanez:

$$\left.\begin{array}{l} n=0 \quad\Rightarrow\quad 4<100\\ n=1 \quad\Rightarrow\quad 32<100\\ n=2 \quad\Rightarrow\quad 384>100\end{array}\right\} \quad\Rightarrow\quad e^{-1/4}\simeq\sum_{n=0}^{2}\frac{(-1)^n}{4^n\cdot n!} \quad\Rightarrow\quad \text{3 batugai hartu behar dira.}$$

5.- Aurkitu hurrengo funtzioen berretura-seriezko garapena, non balio duten adieraziz:

a) $f(x) = \text{L}\left(\dfrac{1+x}{1-x}\right)$

b) $g(x) = \text{L}\left(\sqrt{\dfrac{1+x}{1-x}}\right)$

(1.25 puntu)

a) $f(x) = \text{L}\left(\dfrac{1+x}{1-x}\right) \quad \Rightarrow$

$$\Rightarrow \quad f'(x) = \frac{\dfrac{1-x+1+x}{(1-x)^2}}{\dfrac{1+x}{1-x}} = \frac{2}{(1-x)(1+x)} = \frac{2}{1-x^2} \overset{(1)}{=} \sum_{n=0}^{\infty} 2\left(x^2\right)^n = \sum_{n=0}^{\infty} 2 \cdot x^{2n}$$

(1) Serie geometrikoaren batura, $r = x^2$ delarik.

Beraz, konbergentea da $\Leftrightarrow$ $|r| = \left|x^2\right| = x^2 < 1 \quad \Leftrightarrow \quad |x| < 1$.

Orduan, $f'(x) = \sum_{n=0}^{\infty} 2 \cdot x^{2n} \quad \forall x \in (-1,1)$

Integratuz:

$$f(x) \overset{(2)}{=} \sum_{n=0}^{\infty} 2 \cdot \frac{x^{2n+1}}{2n+1} \quad \forall x \in (-1,1)$$

(2) $\left.\begin{array}{l} \int f'(x)dx = f(x) + K \\ f(0) = \text{L}1 = 0 \end{array}\right\} \Rightarrow \quad K = 0$

Orain, tarteko muturretan aztertuz:

$x = -1$ puntuan $\nexists f$

$x = 1$ puntuan $\nexists f$

Beraz, $f(x) = \sum_{n=0}^{\infty} 2 \cdot \dfrac{x^{2n+1}}{2n+1} \quad \forall x \in (-1,1)$

b) $g(x) = \text{L}\left(\sqrt{\dfrac{1+x}{1-x}}\right) = \dfrac{1}{2}\text{L}\left(\dfrac{1+x}{1-x}\right) = \dfrac{1}{2} f(x) = \dfrac{1}{2}\sum_{n=0}^{\infty} 2 \cdot \dfrac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{2n+1} \quad \forall x \in (-1,1)$

6.- Aurkitu analitiko eta grafikoki $f(x,y)=\arcsin(x-1)+\mathrm{L}\left(2y-y^2\right)+\sqrt{\dfrac{xy-1}{2-xy}}$ **funtzioaren definizio-eremua.**

(1.5 puntu)

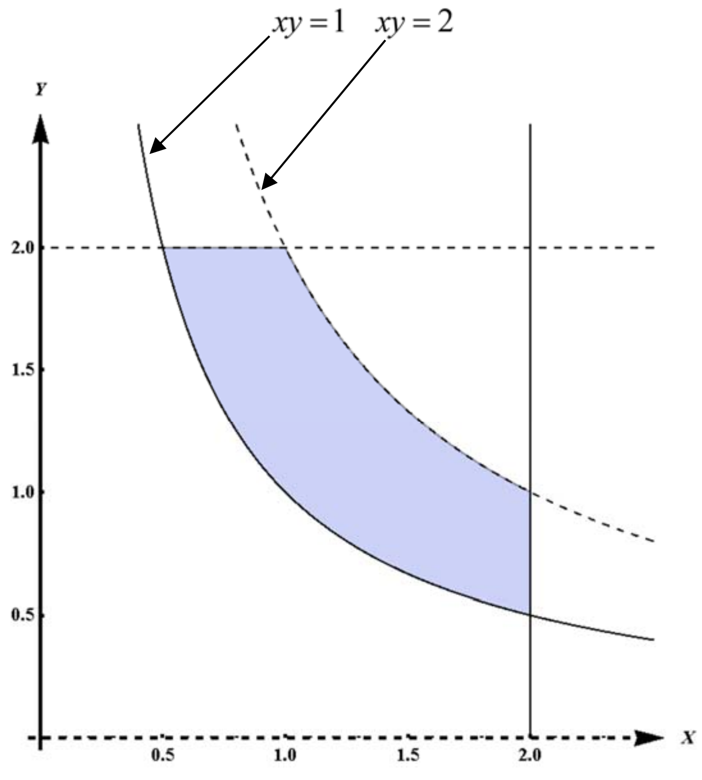
$$D=\left\{(x,y)\in\mathbb{R}^2 \,/\, -1\le x-1\le 1,\ 2y-y^2>0,\ \frac{xy-1}{2-xy}\ge 0,\ 2-xy\ne 0\right\}$$

$$-1\le x-1\le 1 \quad\Leftrightarrow\quad 0\le x\le 2$$

$$2y-y^2>0 \quad\Leftrightarrow\quad y(2-y)>0 \quad\Leftrightarrow\quad \begin{cases} y>0 \text{ eta } 2-y>0 \quad\Leftrightarrow\quad 0<y<2\\ \text{edo}\\ y<0 \text{ eta } 2-y<0 \quad\Leftrightarrow\quad y<0 \text{ eta } y>2 \text{ (ezinezkoa)}\end{cases}$$

$$2-xy\ne 0 \quad\Leftrightarrow\quad xy\ne 2$$

$$\frac{xy-1}{2-xy}\ge 0 \quad\Leftrightarrow\quad \begin{cases} xy-1\ge 0 \text{ eta } 2-xy>0 \quad\Leftrightarrow\quad 1\le xy<2\\ \text{edo}\\ xy-1\le 0 \text{ eta } 2-xy<0 \quad\Leftrightarrow\quad xy\le 1 \text{ eta } xy>2 \text{ (ezinezkoa)}\end{cases}$$



7.- $f(x,y)=\begin{cases} \dfrac{x^3+y^3}{x^2+y^2+y^4} & \forall (x,y)\neq(0,0) \\ A & (x,y)=(0,0) \end{cases}$ **funtzioa emanik,**

a) Aurkitu $A\in\mathbb{R}$ parametroaren balioa, *f* jarraitua izan dadin (0,0) puntuan.

Aurreko atalean lortutako *A* parametroaren baliorako,

b) Kalkulatu *f* –ren deribatu partzialak (0,0) puntuan.

c) Aztertu *f* –ren diferentziagarritasuna (0,0) puntuan.

(1.25 puntu)

a) $$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y^2+y^4}\overset{(1)}{=}\lim_{\substack{\rho\to0^+\\ \forall\theta}}\frac{\rho^3\left(\cos^3\theta+\sin^3\theta\right)}{\rho^2+\rho^4\cdot\sin^4\theta}=$$

$$=\lim_{\substack{\rho\to0^+\\ \forall\theta}}\frac{\rho\left(\cos^3\theta+\sin^3\theta\right)}{1+\rho^2\cdot\sin^4\theta}=\frac{0}{1+0}=0=f(0,0)=A \quad\Leftrightarrow\quad A=0$$

Beraz, *f* jarraitua da (0,0) puntuan $\quad\Leftrightarrow\quad A=0$

(1) polarretan adieraziz: $\begin{cases} x=\rho\cdot\cos\theta \\ y=\rho\cdot\sin\theta \end{cases} \quad \forall\theta\in[0,2\pi)$

b) $$f'_x(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{h^3}{h^2}}{h}=1$$

$$f'_y(0,0)=\lim_{k\to0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to0}\frac{\frac{k^3}{k^2+k^4}}{k}=\lim_{k\to0}\frac{k^2}{k^2+k^4}=\lim_{k\to0}\frac{1}{1+k^2}=\frac{1}{1+0}=1$$

c) Diferentziagarritasuna aztertzeko BBN erabiliko dugu:

f diferentziagarria (0,0) puntuan $\quad\Leftrightarrow\quad \displaystyle\lim_{(h,k)\to(0,0)}\frac{\left|f(h,k)-f(0,0)-h\cdot f'_x(0,0)-k\cdot f'_y(0,0)\right|}{\sqrt{h^2+k^2}}=0$

$$\lim_{(h,k)\to(0,0)}\frac{\left|f(h,k)-f(0,0)-h\cdot f'_x(0,0)-k\cdot f'_y(0,0)\right|}{\sqrt{h^2+k^2}}=\lim_{(h,k)\to(0,0)}\frac{\left|\frac{h^3+k^3}{h^2+k^2+k^4}-h-k\right|}{\sqrt{h^2+k^2}}\overset{(2)}{=}$$

$$\overset{(2)}{=}\lim_{(h,k)\to(0,0)}\frac{\left|\frac{\rho^3\left(\cos^3\theta+\sin^3\theta\right)}{\rho^2+\rho^4\cdot\sin^4\theta}-\rho\cdot\cos\theta-\rho\cdot\sin\theta\right|}{\rho}=$$

$$=\lim_{(h,k)\to(0,0)}\frac{\left|\frac{\rho\left(\cos^3\theta+\sin^3\theta\right)}{1+\rho^2\cdot\sin^4\theta}-\rho\cdot\cos\theta-\rho\cdot\sin\theta\right|}{\rho}=$$

$$= \lim_{(h,k)\to(0,0)} \frac{\left|\rho\left(\cos^3\theta+\sin^3\theta\right)-\left(1+\rho^2\cdot\sin^4\theta\right)\rho\left(\cos\theta+\sin\theta\right)\right|}{\rho\left(1+\rho^2\cdot\sin^4\theta\right)} =$$

$$= \lim_{(h,k)\to(0,0)} \frac{\left|\left(\cos^3\theta+\sin^3\theta\right)-\left(1+\rho^2\cdot\sin^4\theta\right)\left(\cos\theta+\sin\theta\right)\right|}{1+\rho^2\cdot\sin^4\theta} =$$

$$= \left|\left(\cos^3\theta+\sin^3\theta\right)-\left(\cos\theta+\sin\theta\right)\right| \neq 0$$

Beraz, f ez da diferentziagarria (0,0) puntuan.

(2) polarretan adieraziz: $\begin{cases} h=\rho\cdot\cos\theta \\ k=\rho\cdot\sin\theta \end{cases} \quad \forall\theta\in[0,2\pi)$

8.- Kalkulatu $f(x,y)=\begin{cases} \dfrac{1-e^{2x-y}}{y-2x} & \forall(x,y)\,/\,y\neq 2x \\ e^y & \forall(x,y)\,/\,y=2x \end{cases}$ **funtzioaren deribatu partzialak (0,0) puntuan.**

(0.75 puntu)

$$f'_x(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{1-e^{2h}}{-2h}-1}{h}=\lim_{h\to0}\frac{1-e^{2h}+2h}{-2h^2}\overset{(L'H)}{=}\lim_{h\to0}\frac{-2e^{2h}+2}{-4h}\overset{(L'H)}{=}\lim_{h\to0}\frac{-4e^{2h}}{-4}=1$$

$$f'_y(0,0)=\lim_{k\to0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to0}\frac{\frac{1-e^{-k}}{k}-1}{k}=\lim_{k\to0}\frac{1-e^{-k}-k}{k^2}\overset{(L'H)}{=}\lim_{k\to0}\frac{e^{-k}-1}{2k}\overset{(L'H)}{=}\lim_{k\to0}\frac{-e^{-k}}{2}=-\frac{1}{2}$$

9.- Definitu $z=f(x,y)$ **funtzio egokia, eta, bere diferentziala erabiliz, kalkulatu** $\left(0.99\cdot e^{0.02}\right)^5$ **-aren balio hurbildua.**

(Puntu 1)

Izan bedi $z=f(x,y)=\left(x\cdot e^y\right)^5$. Honela, $\left(0.99\cdot e^{0.02}\right)^5 = f(0.99,0.02)$

Bestalde, f funtzio diferentziagarria denez $\forall(x,y)\in\mathbb{R}^2$, orduan $\nabla f \simeq df$ hurbilketa egiaztatzen da, x eta y aldagaien aldakuntzak txikiak direnean. Kasu honetan:

$$\nabla f = f(0.99,0.02)-f(1,0)\simeq df(1,0)=f'_x(1,0)dx+f'_y(1,0)dy$$

non $f(1,0)=1$, $dx=-0.01$ eta $dy=0.02$

$$z=f(x,y)=\left(x\cdot e^y\right)^5=x^5\cdot e^{5y} \Rightarrow \begin{cases} f'_x=5x^4\cdot e^{5y} \Rightarrow f'_x(1,0)=5 \\ f'_y=5x^5\cdot e^{5y} \Rightarrow f'_y(1,0)=5 \end{cases} \Rightarrow df(1,0)=0.05$$

Beraz, $f(0.99,0.02)\simeq f(1,0)+df(1,0)=1.05$