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Application of the Thermal Quadrupoles Method to Semitransparent Solids

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Abstract In this study, the thermal quadrupoles method is extended to semitransparent layered solids. Using this method, the surface temperature of semitransparent multilayered materials is calculated as a function of the optical and thermal properties of each layer. This result eventually leads to determination of the thermal diffusivity, thermal resistance, and/or optical absorption coefficient of layered materials using photothermal techniques. The thermal quadrupoles method is applied to determine the thermal contact resistance in glass stacks.

Keywords Absorption coefficient · Photothermal radiometry · Thermal diffusivity · Thermal quadrupoles

1 Introduction

The thermal quadrupoles method is a simple and compact method that allows calculation of the surface temperature of multilayered opaque materials if the fluxes at the surfaces are known [1]. The relation between temperature and heat flux at both surfaces

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of each layer is expressed in a matrix form that contains the thermal and geometrical properties of the layer. The sample surface temperatures are obtained as a function of the thermal parameters of each layer by computing the product of matrices representative of each layer and, eventually, the corresponding transfer matrices between layers. This method has been successfully applied to characterize opaque materials with in-depth varying thermal conductivities, like case hardened steels [2–4].

The aim of this study is to extend the thermal quadrupoles method to calculate the surface temperature of layered semitransparent materials. It is intended to be applied to materials with in-depth varying thermal and optical properties, as is the case of partially cured resins [5,6], functionally graded materials [7], and multilayered optical systems [8]. As an experimental application, we have retrieved the thermal contact resistance in glass stacks using photothermal radiometry (PTR).

2 Theory

We consider a semitransparent multilayered sample illuminated by a plane light beam (Fig. 1), modulated at a frequency $f\,(\omega = 2\pi f)$. Each layer is characterized by its thermal conductivity (K_i), thermal diffusivity (D_i), optical absorption coefficient (α_i), and thickness (L_i). We first relate the temperature and heat flux at the front and back surfaces of layer i. The one-dimensional heat diffusion equation for each layer is expressed as

$$\frac{\mathrm{d}^2 T_i}{\mathrm{d}z^2} - q_i^2 T_i = -\frac{Q_i}{K_i} \tag{1}$$

where $q_i = \sqrt{\mathrm{i}\omega/D_i}$ is the thermal wave vector and $Q_i = \frac{I_i}{2}\alpha_i \mathrm{e}^{-\alpha_i(z-z_i)}$ is the heat source. I_i is the incident light intensity reaching layer i, which is given by $I_i = I_1 \mathrm{e}^{-(\alpha_1 L_1 + \alpha_2 L_2 + \cdots + \alpha_{i-1} L_{i-1})}$ in the absence of internal reflections. The solution of Eq. 1 is usually expressed in terms of exponential functions [9]. However, hyperbolic

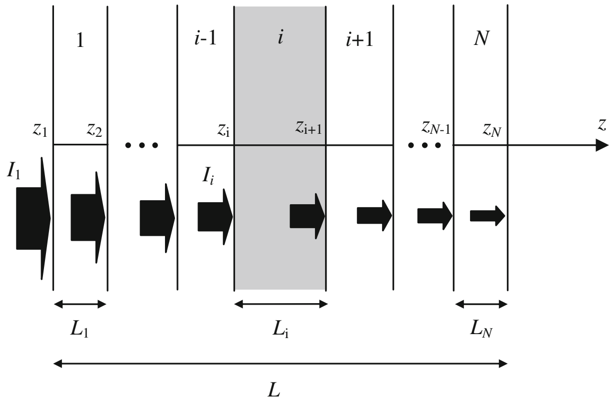

Fig. 1 Diagram of a semitransparent multilayered slab

functions are more convenient when dealing with the thermal quadrupoles method:

$$T_i(z) = A_i \sinh\left[q_i(z-z_i)\right] + B_i \cosh\left[q_i(z-z_i)\right] + C_i \mathrm{e}^{-\alpha_i(z-z_i)}, \tag{2a}$$

$$\begin{aligned}\varphi_i(z) = -K_i \frac{\mathrm{d}T_i}{\mathrm{d}z} &= -K_i q_i \left\{A_i \cosh\left[q_i(z-z_i)\right] + B_i \sinh\left[q_i(z-z_i)\right]\right\}\\ &\quad + K_i C_i \alpha_i \mathrm{e}^{-\alpha_i(z-z_i)},\end{aligned} \tag{2b}$$

where φ is the heat flux.

Applying Eq. 2 at the front (z_i) and rear (z_{i+1}) surfaces of layer i, a matrix relation between temperature and heat flux at both surfaces can be obtained:

$$\begin{pmatrix} T_i(z_i) \\ \varphi_i(z_i) \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} T_i(z_{i+1}) - X_i \\ \varphi_i(z_{i+1}) - Y_i \end{pmatrix}, \tag{3}$$

where

$$a_i = d_i = \cosh(q_i L_i), \quad b_i = \frac{\sinh(q_i L_i)}{K_i q_i}, \quad c_i = K_i q_i \sinh(q_i L_i),$$
$$X_i = C_i \left[\frac{\alpha_i}{q_i} \sinh(q_i L_i) - \cosh(q_i L_i) + \mathrm{e}^{-\alpha_i L_i}\right]$$

and $Y_i = C_i K_i \left[q_i \sinh(q_i L_i) - \alpha_i \cosh(q_i L_i) + \alpha_i \mathrm{e}^{-\alpha_i L_i}\right]$.

Equation 3 generalizes the equation of an opaque layer ($\alpha_i \to \infty$) for which $X_i = Y_i = 0$ [1].

Equation 3 can be rewritten as

$$H_i = M_i (O_i - P_i) \tag{4}$$

where H_i is the input matrix, O_i is the output matrix, M_i is the thermal matrix, and P_i is the optical matrix. To obtain a single matrix equation relating the temperature and heat flux at the front ($z = z_1 = 0$) and rear ($z = z_{N+1} = L$) surfaces of the whole multilayered sample, we need to know the relationship between the temperature and heat flux at each intermediate boundary. Two possibilities are considered:

(a) If there is perfect thermal contact between the layers, the temperature and heat flux continuity can be applied: $T_i(z_{i+1}) = T_{i+1}(z_{i+1})$ and $\varphi_i(z_{i+1}) = \varphi_{i+1}(z_{i+1})$, and therefore $O_i = H_{i+1}$. By applying this equation to each layer, we obtain

$$H_1 = Z_N O_N - \sum_{p=1}^{N} Z_p P_p, \tag{5}$$

where $Z_p = \prod_{i=1}^{p} M_i$, $H_1 = \begin{pmatrix} T_1(0) \\ \varphi_1(0) \end{pmatrix}$, and $O_N = \begin{pmatrix} T_N(L) \\ \varphi_N(L) \end{pmatrix}$. If heat losses are negligible, $\varphi_1(0) = \varphi_N(L) = 0$.

(b) A thermal resistance $R_{i,i+1}$ is introduced to account for the lack of adherence between layers i and $i+1$. This means that the heat flux continuity still holds but there is a jump in temperature given by $T_i(z_{i+1}) = T_{i+1}(z_{i+1}) + R_{i,i+1}\varphi_{i+1}(z_{i+1})$, and therefore $O_i = W_{i,i+1} H_{i,i+1}$, where $W_{i,i+1} = \begin{pmatrix} 1 & R_{i,i+1} \\ 0 & 1 \end{pmatrix}$. This means that the matrix equation relating temperature and heat flux at the front ($z = 0$) and rear ($z = L$) surfaces is similar to Eq. 5:

$$H_1 = Z'_N O_N - \sum_{p=1}^{N} Z'_p P_p \tag{6}$$

with $Z'_p = M_1 W_{1,2} M_2 W_{2,3} M_3 \ldots M_{p-1} W_{p-1,p} M_p$.

3 Applications and Discussion

In order to compare experimental results with theoretical predictions, data need to be normalized so that the frequency dependence of the experimental apparatus is removed. In this study, we use self-normalization (rear-to-front surface signal ratio) because it is more convenient than normalization with a reference sample. We have applied the thermal quadrupoles method to characterize the thermal contact resistance between layers. Experiments have been performed using a phothermal radiometry (PTR) setup described in Ref. [10]. In Fig. 2, we show by symbols the self-normalized PTR signal corresponding to a two-layer sample made of two neutral density filters of the same thickness, $L = 1.04$ mm and properties as follows: $D = 0.54\,\text{mm}^2\cdot\text{s}^{-1}$, $\alpha = 2100\,\text{m}^{-1}$. In order to vary the thermal contact resistance, we have placed a plastic layer with a centered hole of 2 cm diameter between the two glasses. The thicknesses of the films were: 0 (no plastic film), 25 μm, 50 μm, and 75 μm. Two clips were used to press the system. The continuous lines correspond to the simultaneous fit of

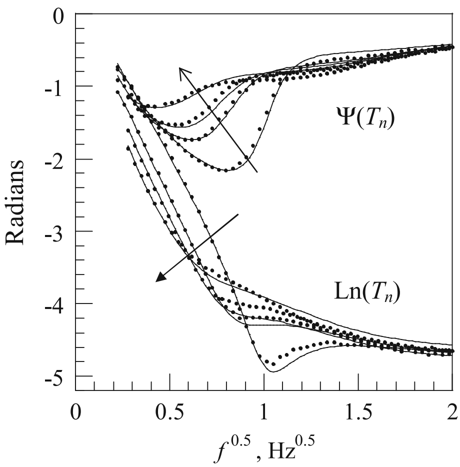

Fig. 2 Modulated PTR measurements of the self-normalized temperature as a function of $\sqrt{f}$ for a filter stack made of two equal neutral density filters ($L = 1.04$ mm) with a holed plastic film as a barrier. The thickness of the plastic film is varied: 0 (no plastic), 25 μm, 50 μm, and 75 μm. The *arrows* indicate increasing thicknesses of the plastic films. *Continuous lines* are the fits with Eq. 6

$\ln(T_n)$ and $\Psi(T_n)$ to Eq. 5 with the thermal resistance R as the fitting parameter. As can be seen, the quality of the fit is good, and the resulting thermal resistances are: $(1.05 \times 10^{-4}, 7.6 \times 10^{-4}, 1.6 \times 10^{-3}$, and $2.3 \times 10^{-3})\ \mathrm{K \cdot m^2 \cdot W^{-1}}$. According to the expression $R = L/K_{\mathrm{air}}$, where $K_{\mathrm{air}} = 0.026\ \mathrm{W \cdot m^{-1} \cdot K^{-1}}$, these thermal resistances correspond to air layers of thickness $2.7\ \mu\mathrm{m}$ (no plastic film), $20\ \mu\mathrm{m}$, $41\ \mu\mathrm{m}$, and $60\ \mu\mathrm{m}$, which are slightly below the geometrical values. This underestimation could be due to a real reduction of the air layer since the clamping decreases the plastic film thickness. Another source of error is that the multiple reflections of the incident light are not included in the model. On the other hand, the neutral filters are completely opaque to wavelengths above $5\ \mu\mathrm{m}$. In order to avoid the effects of the transparency of the sample to IR wavelengths, a spectral IR filter ($5\ \mu\mathrm{m}$ to $12\ \mu\mathrm{m}$) has been placed in front of the detector.

In conclusion, we have extended the thermal quadrupoles method to semitransparent multilayered structures. As a practical application, the ability of the method to assess thermal resistances has been validated experimentally by evaluating the air thickness between two glasses from PTR data. The good agreement between the retrieved air gap thicknesses and the actual distance between glasses confirms the validity of the model. These results open the possibility of applying the thermal quadrupoles method to retrieve in-depth varying optical and thermal properties of heterogeneous samples like dental resins or functionally graded materials.

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References

1. D. Maillet, S. André, J.C. Batsale, A. Degiovanni, C. Moyne, *Thermal Quadrupoles* (Wiley, New York, 2000)
2. R. Celorrio, E. Apiñaniz, A. Mendioroz, A. Salazar, A. Mandelis, J. Appl. Phys. **107**, 083519 (2010)
3. E. Apiñaniz, A. Mendioroz, A. Salazar, R. Celorrio, J. Appl. Phys. **108**, 064905 (2010)
4. R. Celorrio, A. Mendioroz, E. Apiñaniz, A. Salazar, C. Wang, A. Mandelis, J. Appl. Phys. **105**, 083517 (2009)
5. P. Martínez-Torres, A. Mandelis, J.J. Alvarado-Gil, J. Appl.Phys. **106**, 114906 (2009)
6. P. Martínez-Torres, A. Mandelis, J.J. Alvarado-Gil, J. Appl. Phys. **108**, 054902 (2010)
7. Y. Nagasaka, T. Sato, T. Ushiku, Meas. Sci. Technol. **12**, 2081 (2001)
8. J. Ravi, Y. Lu, S. Longuemart, S. Paoloni, H. Pfeiffer, J. Thoen, C. Glorieux, J. Appl. Phys. **97**, 014701 (2005)
9. A. Rosencwaig, A. Gersho, J. Appl. Phys. **47**, 64 (1976)
10. R. Fuente, E. Apiñaniz, A. Mendioroz, A. Salazar, J. Appl. Phys. **110**, 0033515 (2011)