Variational PINNs for solving parametric PDEs

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Abstract:
Solving inverse problems with uncertainty quantification is crucial for our society. Herein, we focus on inverse problems governed by second-order linear Partial Differential Equations (PDEs).

The most critical step for decoding inverse problems is to solve the associated parametric PDE. This step is often a bottleneck when employing classical finite element or finite difference-based methods; however, Neural Networks (NNs) show a promising avenue for solving parametric PDEs.

In this presentation, we use Robust Variational Physics Informed Neural Networks (RVPINNs) [2]. We propose to decompose the parametric PDE solution using a Proper Generalized Decomposition (PGD) [1] of the form:

$$u(x;\xi) \approx \sum_{i=1}^{N} c_i(\xi) \cdot u_i(x), \tag{1}$$

where $\{u_i(x)\}_{i=1}^N$ is a set of functions depending on the spatial variable $x \in \mathbb{R}^d$ (d is the space dimension), and $\{c_i(\xi)\}_{i=1}^N$ is a set of coefficients depending on the PDE parameter ξ. The key idea is to employ a NN $\mathbf{u} : \mathbb{R}^d \longrightarrow \mathbb{R}^N$ to construct the basis functions $u_i(x)$ while computing the coefficients $c_i(\xi)$ using a Least-Squares solver. The output $\mathbf{u}$ plays the role of a set of reduced order basis, as in PGD.

Critical numerical aspects that need to be properly adjusted to ensure the proper behavior of the method include: (a) numerical integration, possibly using a high-order, stochastic, and unbiased rule; (b) a proper architecture choice possibly adapted to the regularity of the solution; and (c) an efficient implementation for the construction of the Least Squares system. We will discuss all these aspects during the presentation.

We will also illustrate the main features and limitations of the method via one- and two-dimensional numerical experiments using a specific RVPINN method known as Deep Fourier Residual [3].

References:

[1] Chinesta, F., Ladeveze, P. and Cueto, E., 2011. A short review on model order reduction based on Proper Generalized Decomposition. Archives of Computational Methods in Engineering, 18(4), pp.395-404.

[2] Rojas, S., Maczuga, P., Muñoz-Matute, J., Pardo, D. and Paszyński, M., 2024. Robust Variational Physics-Informed Neural Networks. Computer Methods in Applied Mechanics and Engineering, 425, p.116904.

[3] Taylor, J.M., Pardo, D. and Muga, I., 2023. A Deep Fourier Residual method for solving PDEs using neural networks. Computer Methods in Applied Mechanics and Engineering, 405, p.115850.